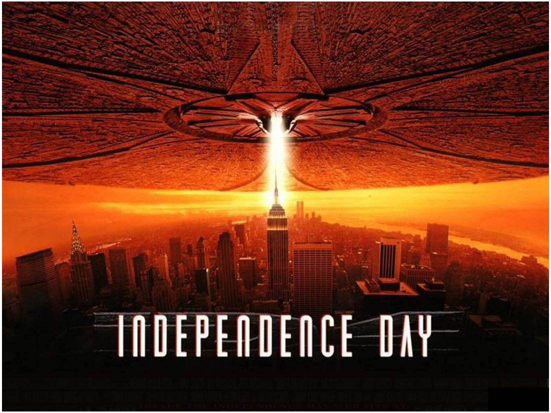
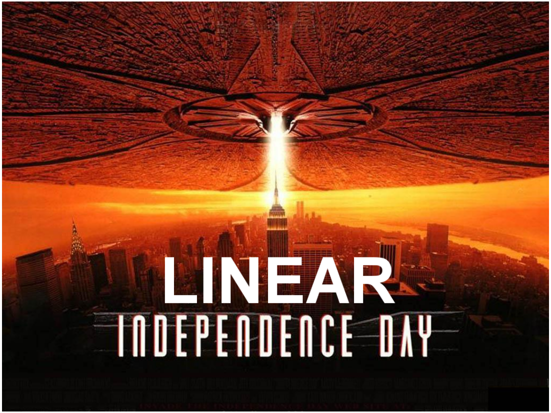
Lecture 05



Math 22 Summer 2017 Section 2
June 30, 2017

Lecture 05



Math 22 Summer 2017 Section 2
June 30, 2017

Just for today

- Finish up §1.5
- Part of §1.6 on network flows
- §1.7 Linear Independence

§1.5 Example

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We now do an example to illustrate the next Theorem.

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$$\begin{bmatrix} 1 & 0 & -3 & 0 & -1 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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How many free variables do we have in this case? As we've seen before with a single free variable, we can write a general solution to this system using a **parametric vector equation**...

§1.5 Example

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$$\mathbf{x} = \begin{bmatrix} 3x_3 + x_5 \\ -2x_3 + 3x_5 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

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What is the geometric interpretation of the solution set?

Now let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and consider the *nonhomogeneous linear system* $A\mathbf{x} = \mathbf{b}$.

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§1.5 Theorem 6

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Thus $\mathbf{v}_h := \mathbf{w} - \mathbf{p}$ satisfies $A\mathbf{x} - \mathbf{0}$ and $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$. This shows that $\mathbf{w} \in S$ and concludes the proof. □

§1.6 Network flows

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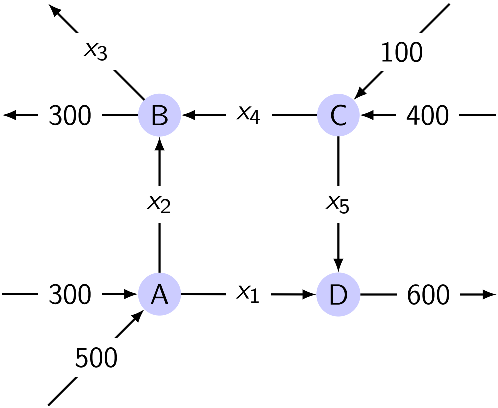

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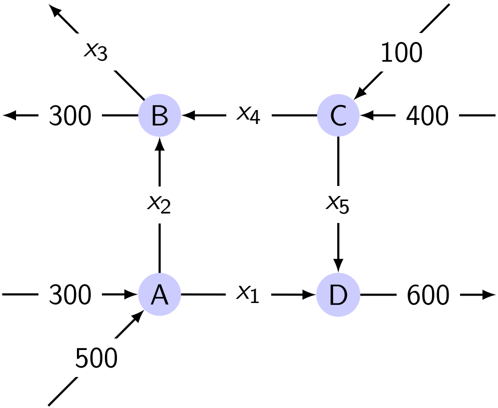

Can you see how this defines a linear system?

§1.6 Network flows

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Now we can use linear algebra to answer questions about the network!

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$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 400 \\ 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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So a general solution to this linear system is given by

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When dealing with network flows, a general solution of this form is called a **general flow pattern** for the network.

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So we see that in this application x_5 is not quite *free*.

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§1.7 Linear independence and $A\mathbf{x} = \mathbf{0}$

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Let's do an example with 4 vectors in $\mathbb{R}^3$.

§1.7 Theorem 8

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Proof.

See previous slide. □

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Could all of the scalars on the RHS of the equation be zero? No! Since we assumed $\mathbf{v}_j \neq 0$. Thus, dividing by c_j we get $\mathbf{v}_j$ as a linear combination of the other vectors.

§1.7 Theorem 7

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Note that a set with one vector $\{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$.

Suppose $\mathbf{v} \neq \mathbf{0}$ and we want to find a vector $\mathbf{w}$ so that $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent.

We summarize our characterization of linear dependence in the following theorem.

Theorem

A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others. Said another way, if and only if at least one of the vectors is in the span of the others.

Proof.

See previous slide. □

Note that a set with one vector $\{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$.

Suppose $\mathbf{v} \neq \mathbf{0}$ and we want to find a vector $\mathbf{w}$ so that $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent. By the theorem, such a $\mathbf{w}$ cannot be in the span of $\mathbf{v}$ which is just all scalar multiples of $\mathbf{v}$.

§1.7 Classwork

Find all values of $h \in \mathbb{R}$ for which the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$$

form a linearly dependent set.

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So the set is dependent if and only if $h = 6$.

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Let A be a $m \times n$ matrix with the property that for every $\mathbf{b} \in \mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

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Let A be a $m \times n$ matrix with the property that for every $\mathbf{b} \in \mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Show that the columns of A are linearly independent.

§1.7 Classwork

If $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$, is $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\}$?

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Yes! Although this seems like a trivial result, the significance is that if we have a space that is spanned by vectors, we can eliminate redundant vectors until we have a linearly independent set. Such a set is called a **basis**.