Lecture 04

Math 22 Summer 2017 Section 2
June 28, 2017

Just for today

- §1.4 Matrix equations
- §1.5 More on solution sets

§1.4 Definition of $A\mathbf{x}$

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Examples? What is required for this definition to make sense? Where does the vector $A\mathbf{x}$ live?

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$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} -10 & -2 & 0 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \\ &= 3 \begin{bmatrix} -10 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 3(-10) + (-1)(-2) + 4(0) \\ 3(0) + (-1)(1) + 4(5) \end{bmatrix} \end{aligned}$$

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§1.4 Theorem 4

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This theorem says that a matrix A either has all of these properties or none of these properties.

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- Careful about $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \neq A$
- The first 3 statements follow directly from the definitions and Theorem 3 (try to convince yourself of this!), so it suffices to show the last statement about pivots is equivalent to any of the others.
- The RREF of A does not depend on the vector $\mathbf{b}$ in the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$.

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$(d) \implies (a)$: Suppose (d) is true.

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Example of theorem in use?

§1.4 Theorem 5

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Try to prove these as an exercise. □

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Where have you seen linear maps before?

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$$T(f+g) = (f+g)' = f' + g' = T(f) + T(g)$$
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The study of linear maps given by matrices is of primary importance in linear algebra.

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§1.5 Homogeneous linear systems

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A linear system is **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$ with $A_{m\times n}$ (notation for $m \times n$ matrix) and $\mathbf{0} \in \mathbb{R}^m$.

What can you observe about the solution set of a homogeneous system? It always has the **trivial solution** of $\mathbf{x} = \mathbf{0}$. Where does this $\mathbf{0}$ live?

How can we tell if a homogeneous system has a nontrivial solution? Well, we know that a consistent system has a unique solution or infinitely many solutions. When do we get infinitely many solutions? When we have free variables. When do we have free variables? When the number of pivots is strictly less than the number of variables.

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To summarize, we have the following theorem...

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Proof.

This follows from discussion on previous slide: since the number of pivots is $\leq m < n$ and the number of variables is n, we see that we must have free variables in this case. $\square$

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Suppose now we choose a specific A in RREF given by

$$\begin{bmatrix} 1 & 0 & -3 & 0 & -1 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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How many free variables do we have in this case? As we've seen before with a single free variable, we can write a general solution to this system using a **parametric vector equation**...

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$$\mathbf{x} = \begin{bmatrix} 3x_3 + x_5 \\ -2x_3 + 3x_5 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

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What is the geometric interpretation of the solution set?

Now let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and consider the *nonhomogeneous linear system* $A\mathbf{x} = \mathbf{b}$.

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§1.5 Theorem 6

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Suppose that the equation $A\mathbf{x} = \mathbf{b}$ *is consistent for some given* $\mathbf{b}$*, and let* $\mathbf{p}$ *be a particular solution.*

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Let S be the set of solutions to $A\mathbf{x} = \mathbf{b}$. Let $T = \{\mathbf{p} + \mathbf{v}_h : \mathbf{v}_h \text{ satisfies } A\mathbf{x} = \mathbf{0}\}$.

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$(T \subseteq S)$:
An arbitrary element of T is of the form $\mathbf{p} + \mathbf{v}_h$. But $A\mathbf{p} = \mathbf{b}$ and $A\mathbf{v}_h = \mathbf{0}$. How do we conclude that $A(\mathbf{p} + \mathbf{v}_h) = \mathbf{b}$?

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