

Lecture 02

Math 22 Summer 2017 Section 2 June 24, 2017

§1.2 Row Reduction and Echelon Forms





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We call this process row reduction or Gaussian elimination.

§1.2 Leading entries of a matrix





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- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zero.





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- The leading entry in each nonzero row is a 1.
- Each leading 1 is the only nonzero entry in its column.

§1.2 Pivots





A **pivot position** in a matrix A is a location in A corresponding to a leading 1 in the reduced row echelon form of A.



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- A **pivot column** is a column of *A* containing a pivot position.
- A **pivot** is a nonzero entry in a pivot position.

§1.2 Remarks





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- The reduced echelon form of a matrix is unique.
- Although an echelon form of a matrix is not unique, the positions of the leading entries does not change regardless of the echelon form.



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- 4. Consider the submatrix obtained by removing the row and column containing the pivot. Go to the first step and repeat until there is no submatrix left to consider.



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Steps 1 - 4 obtains an echelon form of the original matrix. Step 5 obtains the reduced echelon form of the original matrix.



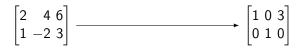
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Steps 1 - 4 obtains an echelon form of the original matrix. Step 5 obtains the reduced echelon form of the original matrix. Let's do a 2×3 example by hand...

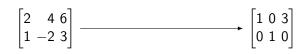
§1.2 Example





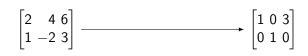






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This corresponds to the linear system

$$\begin{vmatrix} x_1 + 3x_3 = 1 \\ x_2 - 2x_3 = 5 \end{vmatrix} \xrightarrow{x_1 = 1 - 3x_3} \\ x_2 = 5 + 2x_3 \end{vmatrix}$$



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The variables corresponding to pivot columns (i.e x_1 and x_2) are called **basic variables**. The other variables (i.e. x_3) are called **free variables**. We can use this description to write the set of solutions *parametrically*...





The solutions of the linear system on the previous slide can be described as follows.

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This explains the name *parametric* since the solution set is described by the parametric equation of a line in \mathbb{R}^3 .

§1.2 Existence and uniqueness questions



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Theorem

For a consistent system, the solution set is one of the following:

- (a) a unique solution (no free variables)
- (b) infinitely many solutions (at least one free variable)