Lecture 02

Math 22 Summer 2017 Section 2
June 24, 2017

§1.2 Row Reduction and Echelon Forms

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$$\begin{bmatrix} 1 & 0 & 0 & -5 & 4 & 1 & 6 & 1 \\ 0 & 1 & 2 & 5 & 6 & 6 & 8 & 8 \\ 1 & -1 & -1 & -6 & 2 & -2 & 3 & -3 \\ 0 & 1 & 1 & 1 & 3 & 3 & 4 & 5 \\ -1 & 1 & 1 & 6 & -1 & 2 & -2 & 4 \\ 0 & -1 & -2 & -5 & -4 & -6 & -6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -5 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & -3 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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We call this process **row reduction** or **Gaussian elimination**.

§1.2 Leading entries of a matrix

Definition

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- All entries in a column below a leading entry are zero.

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- Each leading 1 is the only nonzero entry in its column.

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- Although an echelon form of a matrix is not unique, the positions of the leading entries does not change regardless of the echelon form.

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Let's do a 2×3 example by hand...

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Computing RREF by hand requires practice (see classwork).

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The variables corresponding to pivot columns (i.e x_1 and x_2) are called **basic variables**. The other variables (i.e. x_3) are called **free variables**. We can use this description to write the set of solutions *parametrically*...

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The solutions of the linear system on the previous slide can be described as follows.

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This explains the name *parametric* since the solution set is described by the parametric equation of a line in $\mathbb{R}^3$.

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with $b \neq 0$.

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Theorem

For a consistent system, the solution set is one of the following:

(a) *a unique solution (no free variables)*

(b) *infinitely many solutions (at least one free variable)*