Lecture 01

Math 22 Summer 2017 Section 2
June 23, 2017

Why Linear Algebra?

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- Linear algebra is ubiquitous in the pure and applied mathematical sciences
- Linear objects are tractable by computer

Goals of this course

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- Provide an introduction to abstract mathematics: definitions, theorems, proofs in the context of linear algebra

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- Introduce the mechanical tools used in linear algebra
- Provide an introduction to abstract mathematics: definitions, theorems, proofs in the context of linear algebra
- Give interesting examples of applications of linear algebra

Math 22 Course Information

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https://math.dartmouth.edu/~m22x17

Definition

A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation

$$a_1 x_1 + \cdots + a_n x_n = b$$

where b and the **coefficients** a_i are real or complex numbers.

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What's an example of a system of equations that is not linear?

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4. same plane in $\mathbb{R}^3$

$$2x_1 + 3x_2 + 5x_3 = 7$$
$$6x_1 + 9x_2 + 15x_3 = 21$$

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From this we extract a **coefficient matrix** and **augmented matrix** given below:

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\begin{bmatrix} 2 & 3 & 5 \\ 2 & 3 & 7 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 5 & 7 \\ 2 & 3 & 7 & 11 \end{bmatrix}
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Here we have a 2×3 matrix and a 2×4 matrix.

As an example, consider the linear system

$$
\begin{aligned}
x_1 - 2x_4 &= -3 \\
2x_2 + 2x_3 &= 0 \\
x_3 + 3x_4 &= 1 \\
-2x_1 + 3x_2 + 2x_3 + x_4 &= 5
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Find the corresponding coefficient matrix and augmented matrix.

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ -2 & 3 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix}$$

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We do this because we want to find *solutions* to linear systems.

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Let's see how to solve the linear system from before:

$$\begin{aligned} x_1 - 2x_4 &= -3 \\ 2x_2 + 2x_3 &= 0 \\ x_3 + 3x_4 &= 1 \\ -2x_1 + 3x_2 + 2x_3 + x_4 &= 5. \end{aligned}$$

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Alternatively, the set of solutions to the linear system is given by

$$\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ -3 \\ 1 \end{bmatrix} : x_4 \in \mathbb{R} \right\} \subset \mathbb{R}^4.$$

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Here we emphasize that solution sets are *sets of vectors*.

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One proves that row operations do not change solution sets.

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The system is consistent and has a unique solution. It is $(x_1, x_2, x_3) = (1, -2, 4)$.

Moreover, any augmented matrix that is row equivalent to the one above corresponds to a linear system with the same unique solution.

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The system has no solutions (inconsistent).

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What can we say about the solutions?

The system has no solutions (inconsistent).

Moreover, any augmented matrix that is row equivalent to the one above corresponds to an inconsistent system.

https://math.dartmouth.edu/~m22x17/sched.html