

Lecture 01

Math 22 Summer 2017 Section 2 June 23, 2017

Why Linear Algebra?





Solutions to differential equations



- Solutions to differential equations
- Applications such as page rank algorithms and data compression



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- Linear algebra is ubiquitous in the pure and applied mathematical sciences



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- Applications such as page rank algorithms and data compression
- Linear algebra is ubiquitous in the pure and applied mathematical sciences
- Linear objects are tractable by computer

Goals of this course





Introduce the mechanical tools used in linear algebra



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- Provide an introduction to abstract mathematics: definitions, theorems, proofs in the context of linear algebra



- Introduce the mechanical tools used in linear algebra
- Provide an introduction to abstract mathematics: definitions, theorems, proofs in the context of linear algebra
- Give interesting examples of applications of linear algebra

Math 22 Course Information





https://math.dartmouth.edu/~m22x17

§1.1 Systems of Linear Equations



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$$a_1x_1+\cdots+a_nx_n=b$$

where b and the **coefficients** a_i are real or complex numbers.

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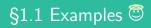
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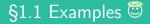
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You've almost certainly encountered linear equations previously. Examples?

What's an example of a system of equations that is not linear?







1. plane in \mathbb{R}^3



 $2x_1 + 3x_2 + 5x_3 = 7$

§1.1 Examples 😇

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2. non parallel planes in
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4. same plane in \mathbb{R}^3

$$2x_1 + 3x_2 + 5x_3 = 7$$

$$6x_1 + 9x_2 + 15x_3 = 21$$



§1.1 Representing linear systems as matrices





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From this we extract a **coefficient matrix** and **augmented matrix** given below:

$$\begin{bmatrix} 2 & 3 & 5 \\ 2 & 3 & 7 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 5 & 7 \\ 2 & 3 & 7 & 11 \end{bmatrix}$$



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Here we have a 2×3 matrix and a 2×4 matrix.

§1.1 Representing linear systems as matrices

As an example, consider the linear system

$$x_1 - 2x_4 = -3$$
$$2x_2 + 2x_3 = 0$$
$$x_3 + 3x_4 = 1$$
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Find the corresponding coefficient matrix and augmented matrix.

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ -2 & 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix}$$



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We do this because we want to find *solutions* to linear systems.

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Let's see how to solve the linear system from before:

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$$3x_{2} + 2x_{3} - 3x_{4} = -1$$

Replace row 4 with row 4 plus twice row 1 ($R_4 \leftarrow R_4 + 2R_1$)

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}$$





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Alternatively, the set of solutions to the linear system is given by

$$\left\{ \begin{bmatrix} -3\\ -1\\ 1\\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2\\ 3\\ -3\\ 1 \end{bmatrix} : x_4 \in \mathbb{R} \right\} \subset \mathbb{R}^4.$$

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Here we emphasize that solution sets are sets of vectors.

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We say two matrices are **row-equivalent** if one is obtained from the other by finitely many row operations.

One proves that row operations do not change solution sets.



$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$



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The system is consistent and has a unique solution.



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The system is consistent and has a unique solution. It is $(x_1, x_2, x_3) = (1, -2, 4)$.



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What can we say about the solutions?

The system is consistent and has a unique solution. It is $(x_1, x_2, x_3) = (1, -2, 4)$.

Moreover, any augmented matrix that is row equivalent to the one above corresponds to a linear system with the same unique solution.



$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$



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The system has no solutions (inconsistent).



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What can we say about the solutions?

The system has no solutions (inconsistent).

Moreover, any augmented matrix that is row equivalent to the one above corresponds to an inconsistent system.



https://math.dartmouth.edu/~m22x17/sched.html