Your name: SOLUTIONS

8/23/17 Barnett

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Math 22 Summer 2017, Homework 9, due start of Wed Aug 23 class

Total pts: 14.

This one is shorter since you have 2 days less.

(1) Linear regression! Let x_1 be the intercept and x_2 be the slope for a general linear function $y(t) = x_1 + x_2 t$. Find its *least squares fit* to the data $(0,0)$, $(2,-2)$, and $(3,4)$, which are three points (t,y) in the plane. Here's how to set up the linear system (you don't need to read Sec. 6.6 unless interested): The first point says $x_1 + x_2.0 = 0$, the next says $x_1 + x_2.2 = -2$, and the last says $x_1 + x_2.3 = 4$.

5 pts. (a) The system is inconsistent. Find the least squares solution vector(s) $\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

Normal eqns need

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 13 \end{bmatrix}, \quad A^T \vec{b} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

Solve $(A^T A)\hat{x} = A^T \vec{b}$:

$$\left[\begin{array}{cc|c} 3 & 5 & 2 \\ 5 & 13 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 5 & 2 \\ 15 & 39 & 24 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 5 & 2 \\ 0 & 14 & 14 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 3 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

So $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

yes, since $A^T A$ full rank. (1 pt)

You've now learned the best-fit intercept $\hat{x}_1$ and slope $\hat{x}_2$! Is this solution unique?

2 pts.

(b) Let A be *any* matrix, possibly rectangular. Prove that if $A^T A$ is invertible, then the columns of A are linearly independent.

Let $\vec{x}$ solve $A\vec{x} = \vec{0}$.

Left-multiply by A^T: $A^T A \vec{x} = A^T \vec{0} = \vec{0}$

By I.M.T., since $A^T A$ is invertible, the homog. lin. sys $(A^T A)\vec{x} = \vec{0}$ has only the soln $\vec{x} = \vec{0}$.

Thus $\vec{x} = \vec{0}$. By defn. of L.I, cols. of A are L.I.

$\square$

(2) (a) Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Over all vectors $\mathbf{x}$ in $\mathbb{R}^3$ with $\|\mathbf{x}\| = 1$, what is the largest $\|A\mathbf{x}\|$ can be? [Hint: if it helps, exploit that that AA^T and $A^T A$ have identical *nonzero* eigenvalues.]

3 pts.

$$\sigma_1 = \max_{\|\vec{x}\|=1} \|A\vec{x}\| \text{ by definition.} = \sqrt{\lambda_1(A^T A)} = \sqrt{\lambda_1(AA^T)}$$

exploiting hint.

$$AA^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

Find eigvals:

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 4 = \lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-6) = 0 \qquad \lambda_1 = 6, \ \lambda_2 = 1$$

the largest.

So $\sigma_1 = \sqrt{\lambda_1} = \sqrt{6}$

4 pts

(b) Compute by hand the full SVD of the previous A, ie give U, Σ, and V. [Hints: find the third column of V however you like, and make sure that your $\mathbf{u}_j$ vectors match your $\mathbf{v}_j$ vectors in ordering and sign]

Right sing. vecs. $\vec{v}_j$ are eigvecs of A^TA corresp. to the λ_j's.

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\boxed{\lambda_1 = 6:} \quad A^TA - 6I = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -5 & 0 \\ 1 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 2 & -4 \\ 0 & -5 & 10 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} & & x_3 \text{ free} \end{array} \\ \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\text{so} \quad \begin{aligned} x_1 &= 5x_3 \\ x_2 &= 2x_3 \\ x_3 &= x_3 \end{aligned} \qquad \vec{v}_1{}' = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize}} \vec{v}_1 = \begin{bmatrix} 5/\sqrt{30} \\ 2/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}$$

$$\boxed{\lambda_2 = 1:} \quad A^TA - I = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so } \vec{v}_2{}' = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

free

$\vec{v}_3$ get since $\{\vec{v}_3\}$ is basis for Nul A: $A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ so $\vec{v}_3{}' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

free

$$\text{So } V = \begin{bmatrix} 5/\sqrt{30} & 0 & -1/\sqrt{6} \\ 2/\sqrt{30} & -1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & 1/\sqrt{6} \end{bmatrix}$$

compute

$$A\vec{v}_1{}' = \begin{bmatrix} 12 \\ 6 \end{bmatrix} \xrightarrow{\text{norm.}} \vec{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \quad A\vec{v}_2{}' = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \xrightarrow{\text{norm.}} \vec{u}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

note easier to use unnormalized $\vec{v}$'s here ☺

So,

σ_1, σ_2

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 5/\sqrt{30} & 2/\sqrt{30} & 1/\sqrt{30} \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}}_{V^T}$$

is full SVD.