Your name:

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Math 22 Summer 2017, Homework 8, due Fri August 18 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

7pts

(1) Let $A = \begin{bmatrix} 5 & 1 & -1 & 5 \\ 1 & 5 & 5 & -1 \\ -5 & 1 & 1 & 5 \end{bmatrix}$ and note that the rows of A are orthogonal.

4pts. (a) Without using row reduction, write $\mathbf{y} = \begin{bmatrix} 11 \\ 5 \\ 3 \\ -1 \end{bmatrix}$ as a linear combination of the rows of A.

A has orthogonal rows $\vec{u}_1, \vec{u}_2, \vec{u}_3$

Thus $\vec{y} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \alpha_3 \vec{u}_3$ where $\alpha_i = \dfrac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}$

$\alpha_1 = \dfrac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \dfrac{52}{52} = 1$, $\alpha_2 = \dfrac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \dfrac{52}{52} = 1$

$\alpha_3 = \dfrac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \dfrac{-52}{52} = -1$

So, $\vec{y} = \vec{u}_1 + \vec{u}_2 - \vec{u}_3$

1pt. (b) Let $W = \text{Col}\,(A^T)$. To which fundamental subspace of the matrix A is $W^\perp$ equal?

$\text{Col}(A^T) = \text{Row}\,A$ so $W^\perp = \text{Nul}\,A$

(recall from §6.1, $(\text{Row}\,A)^\perp = \text{Nul}\,A$).

2pts (c) What is dim $W^\perp$? Prove your answer.

Either: • row reduce A to show 3 pivots, 1 free var, so $\dim \text{Nul}\,A = 1$.

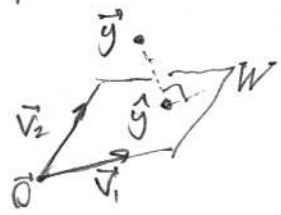
Or: $\underbrace{\dim W}_{3} + \dim W^\perp = n$ (proved in §6.3)

3 since W has an (orthog) basis of 3 els. n: 4 since $\vec{u}_i \in \mathbb{R}^4$.

So $\dim W^\perp = 1$.

[2pts] (2) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$. Let $W = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

3pts (a) Find the element of W whose distance to $\mathbf{y}$ is as small as possible.



let $\hat{y}$ be this element; by best approx. thm,

$$\hat{y} = \mathrm{proj}_W \vec{y}$$

$$= \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \frac{-6}{6}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

1pt. (b) Compute the distance from the previous part.

$$\mathrm{dist}(\vec{y}, W) = \|\vec{y} - \hat{y}\| = \sqrt{(-1-0)^2 + (1-0)^2 + (3-3)^2}$$
$$= \sqrt{2}$$

2pts. (c) Let U be the 3×2 matrix whose columns are $\mathbf{v}_1/\|\mathbf{v}_1\|$ and $\mathbf{v}_2/\|\mathbf{v}_2\|$. *Without* computing any matrix-vector multiplication, find $UU^T\mathbf{y}$ and explain why.

U has orthonormal columns, thus UU^T is the orthogonal projector onto W. Thus $UU^T\vec{y} = \hat{y}$ as found above. So the answer is $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

1pt. (d) Without computing a matrix-vector multiplication, compute the 2-norm of the vector $U\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, with U as in part (c). Explain.

For any matrix U with orthonormal columns, and any vector $\vec{x}$, $\|U\vec{x}\| = \|\vec{x}\|$.

Thus $\left\|U\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right\| = \sqrt{4^2+3^2} = 5.$

6pts

(3) Let $A = \begin{array}{c} \begin{matrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{matrix} \\ \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$.

3pts (a) Find an orthogonal basis for Col A using the Gram-Schmidt algorithm.

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{a}_1 \qquad \text{so } \vec{v}_1 \cdot \vec{v}_1 = \|\vec{v}_1\|^2 = 2.$$

$$\vec{v}_2\,' = \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \overset{\text{rescale}}{\rightsquigarrow} \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{v}_3\,' = \vec{a}_3 - \cancel{\frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}} \vec{v}_1 - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} - \frac{(-4)}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \\ 1 \end{bmatrix}$$

$\rightarrow 0$

$$\underset{\rightsquigarrow}{\text{rescale}}\ \vec{v}_3 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 3 \end{bmatrix}$$

So, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ 3 \end{bmatrix} \right\}$ is orthog basis for Col A.

3pts. (b) Compute the QR factorization of A. (i.e. Find matrices Q and R so that $A = QR$, the columns of Q form an orthonormal basis for Col A, and R is an upper triangular invertible matrix with positive entries along its diagonal.)

$$Q = \text{normalized } \vec{v}_j\text{'s stacked} = \begin{bmatrix} \frac{\vec{v}_1}{\sqrt{2}} & \frac{\vec{v}_2}{\sqrt{6}} & \frac{\vec{v}_3}{\sqrt{21}} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 2/\sqrt{21} \\ 1/\sqrt{2} & 1/\sqrt{6} & 2/\sqrt{21} \\ 0 & 2/\sqrt{6} & -2/\sqrt{21} \\ 0 & 0 & 3/\sqrt{21} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{\frac{3}{2}} & -\frac{4}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{7}{3}} \end{bmatrix}$$

$\leftarrow$ obviously, equivalent forms of these numbers are correct.

$\nearrow$ upper triangular form: if not, lose at least 1pt.

up to +1pt: BONUS what happens during Gram-Schmidt if the columns of A were linearly dependent?

(for $\vec{v}_j = \vec{0}$) One of the resulting $\vec{v}_j$ vectors is $\vec{0}$, and the process breaks down after that since dividing by $\vec{v}_j \cdot \vec{v}_j$ is needed.