Your name:

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Math 22 Summer 2017, Homework 7, due Fri Aug 11

There is limited space, but please show the key intermediate steps to achieve full credit.

[7pts] (1) Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, which you should check maps the vector $\mathbf{x}^{(k)} := \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$ to $\mathbf{x}^{(k+1)} := \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix}$, where f_k, $k = 0, 1, \ldots$, is the Fibonacci sequence 1,1,2,3,5,8,....

[4pt] (a) Write the matrix in the form $A = PDP^{-1}$ (ie give all three matrices). Please use the *golden ratio* $\phi = (1+\sqrt{5})/2$ for working and answers; note $-\phi^{-1} = (1-\sqrt{5})/2$. Please also choose the 2nd row of P to be [1 1]. ← so ordering λ_1, λ_2 could be swapped.

Find eigenvalues: $\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$, $\lambda = \frac{1 \pm \sqrt{5}}{2} = \phi, -\phi^{-1}$

$\lambda_1 = \phi$: $A - \phi I = \begin{bmatrix} 1-\phi & 1 \\ 1 & -\phi \end{bmatrix} \sim \begin{bmatrix} 1 & -\phi \\ 0 & 0 \end{bmatrix}$ (free) so $\vec{v}_1 = \begin{bmatrix} \phi \\ 1 \end{bmatrix}$

$\lambda_2 = -\phi^{-1}$: $A + \phi^{-1} I = \begin{bmatrix} 1+\phi^{-1} & 1 \\ 1 & \phi^{-1} \end{bmatrix}$ so $\vec{v}_2 = \begin{bmatrix} -\phi^{-1} \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} \phi & -\phi^{-1} \\ 1 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\frac{1}{2}+\frac{\sqrt5}{2}+\frac{1}{2}-\frac{\sqrt5}{2}}\begin{bmatrix} 1 & \phi^{-1} \\ -1 & \phi \end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix} 1 & \phi^{-1} \\ -1 & \phi \end{bmatrix}$, $D = \text{diag}\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 0 & -\phi^{-1} \end{bmatrix}$

2pts. (b) Use this to write a formula for $\mathbf{x}^{(k)}$, for the dynamical system with matrix A, for the initial vector $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Reading off the first component gives an explicit formula for the kth Fibonacci number! (Isn't it weird that f_k must always be an integer?)

$\vec{x}^{(k)} = A^k \vec{x}^{(0)} = PD^kP^{-1}\vec{x}^{(0)}$ → $\frac{1}{\sqrt5}\begin{bmatrix} 1 & \phi^{-1} \\ -1 & \phi \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= P\frac{1}{\sqrt5}\begin{bmatrix} \lambda_1^k \\ -\lambda_2^k \end{bmatrix} = \frac{1}{\sqrt5}P\begin{bmatrix} \phi^k \\ -(-\phi)^{-k} \end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix} \phi^{k+1} - (-\phi)^{-k-1} \\ \phi^k - (-\phi)^{-k} \end{bmatrix}$ ← entry f_k check: given

dominate as $k \to \infty$

k	f_k
1	1
2	2
3	3
4	5
5	8

1 pt (c) To what does f_{k+1}/f_k converge, if anything, as $k \to \infty$, and why? (explain using your formula).

Recall $\vec{x}^{(k)} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2$ for general $n=2$

since $\lambda_1 > \lambda_2$ then the 1st term dominates as $k \to \infty$

so $\vec{x}^{(k)}$ converges to the $\vec{v}_1$ direction

$c\begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$ $c\begin{bmatrix} \phi \\ 1 \end{bmatrix}$ $\Rightarrow \lim_{k\to\infty}\frac{f_{k+1}}{f_k} = \phi$

[7pt] (2) You play the following "game," which involves a decisions made on the hour. If you are exercising this hour, then for sure the next hour you will start studying. If you are studying, then at the end of the hour you toss a fair coin: heads you continue studying, tails you switch to exercise. (This basic model doesn't include eating or sleeping. Details!)

[2pts] (a) Let the ordering be $\mathbf{x} = \begin{bmatrix} \text{probability of exercising} \\ \text{probability of studying} \end{bmatrix}$. Write a stochastic matrix A whose Markov chain models the game.

ordering crucial, must match that of $\vec{x}$:

$$\begin{array}{c} \\ E \\ S \end{array} \begin{array}{c} \begin{matrix} E & S \end{matrix} \\ \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} \end{array} = A$$

[1pt] (b) Is the matrix A *regular*? Explain. Even though A has a zero entry, ...

$A^2 = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$ has all entries strictly positive, so A is regular.

[2pts] (c) Say your initial vector is $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To what, if anything, does your probability of studying tend in the long-time limit $k \to \infty$? Explain.

Since A is regular, for <u>any</u> initial vector $\vec{x}^{(0)}$, $\lim_{k\to\infty} \vec{x}^{(k)} = \vec{q}$ where $\vec{q}$ is the <u>unique</u> steady state vector.

Solve for $\vec{q}$: $A\vec{x} = \vec{x}$ so $\vec{x} \in \text{Nul}(A - I) = \text{Nul} \begin{bmatrix} -1 & 1/2 \\ 1 & -1/2 \end{bmatrix}$

so $\vec{x}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\xrightarrow{\text{make total } \sum_j q_j = 1.}$ $\vec{q} = \frac{1}{3}\vec{x}' = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \begin{matrix} \leftarrow E \\ \leftarrow S \end{matrix}$

Your prob. of studying tends to $2/3$.

[2pts] (d) Prove the claim from class that any stochastic matrix A must have an eigenvalue 1. [Hint: find a simple eigenvector of A^T.]

A^T has <u>rows</u> that are probability vectors, so $\sum_{j=1}^{n} (A^T)_{ij} = 1$, for each $i = 1 \cdots n$.

Thus for $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, $A^T\vec{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 1\vec{v}$

thus A^T has an eigenvalue 1.

Since A & A^T have <u>identical</u> eigenvalues, A also has eigenvalue 1.
↳ don't need to prove, but: $\det(A - \lambda I) = \det(A^T - \lambda I)$.

6 pts (3) The web consists of three pages. a lints to b. b links to c. c links to a and b. Use the PageRank algorithm with $\alpha = 1$. ← ie, no random jumps, just the simplest version

3 pts. (a) Write the stochastic matrix then solve for the vector of importances (order it abc and normalize so that the vector sums to one).

$$A = \begin{bmatrix} 0 & 0 & 1/2 \\ 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

we know $\lambda = 1$ is eigenvalue

$$\Rightarrow A - I = \begin{bmatrix} -1 & 0 & 1/2 \\ 1 & -1 & 1/2 \\ 0 & 1 & -1 \end{bmatrix}$$

Ref.:

$$\Rightarrow A - I \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(free) — param. vec. form

$$\vec{x} = \begin{bmatrix} \frac{1}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$$

normalize to total 1

$$\vec{q} = \frac{\vec{x}}{\sum x_i} = \frac{1}{5/2}\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \end{bmatrix}$$

3 pts (b) In a rather selfish effort to increase its rank (importance score), page a removes its link to b. [Hint: make sure you understand how the algorithm deals with such a situation.] What is the new vector of importances? Did the effort work?

by jumping to random place, ie col. vec of all entries $1/n$

$$A = \begin{bmatrix} 1/3 & 0 & 1/2 \\ 1/3 & 0 & 1/2 \\ 1/3 & 1 & 0 \end{bmatrix}$$

$$A - I = \begin{bmatrix} -2/3 & 0 & 1/2 \\ 1/3 & -1 & 1/2 \\ 1/3 & 1 & -1 \end{bmatrix} \text{ scale to integers } \sim \begin{bmatrix} -4 & 0 & 3 \\ 2 & -6 & 3 \\ 1 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/4 \\ 0 & -6 & 9/2 \\ 0 & 6 & -9/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/4 \\ 0 & 1 & -3/4 \\ 0 & 0 & 0 \end{bmatrix}$$

(free)

$$\text{so } \vec{x} = \begin{bmatrix} 3/4 \\ 3/4 \\ 1 \end{bmatrix} \quad \text{divide by total} \rightarrow \quad \vec{q} = \begin{bmatrix} 3/10 \\ 3/10 \\ 2/5 \end{bmatrix}$$

yes, score of a went up from $\frac{2}{10}$ to $\frac{3}{10}$!

up to +2 BONUS If the PageRank Markov chain iteration were used to approximate solutions to (a) and (b), would they converge, and why?

Easiest is to find some power A^k, e.g. in MATLAB

a) $A^5 = \begin{bmatrix} 1/4 & 1/4 & 1/8 \\ 1/4 & 1/2 & 3/8 \\ 1/2 & 1/4 & 1/2 \end{bmatrix}$ all entries are > 0 ⇒ regular ⇒ yes.

b) A^2 has all entries > 0 ⇒ regular ⇒ yes.