

Your name:

Instructor (please circle):

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Math 22 Summer 2017, Homework 7, due Fri Aug 11

There is limited space, but please show the key intermediate steps to achieve full credit.

(1) Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, which you should check maps the vector $\mathbf{x}^{(k)} := \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$

to $\mathbf{x}^{(k+1)} := \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix}$, where $f_k, k = 0, 1, \dots$, is the Fibonacci sequence $1, 1, 2, 3, 5, 8, \dots$.

(a) Write the matrix in the form $A = PDP^{-1}$ (ie give all three matrices). Please use the *golden ratio* $\phi = (1 + \sqrt{5})/2$ for working and answers; note $-\phi^{-1} = (1 - \sqrt{5})/2$. Please also choose the 2nd row of P to be $[1 \ 1]$.

(b) Use this to write a formula for $\mathbf{x}^{(k)}$, for the dynamical system with matrix A , for the initial vector $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Reading off the first component gives an explicit formula for the k th Fibonacci number! (Isn't it wild that f_k must always be an integer?)

(c) To what does f_{k+1}/f_k converge, if anything, as $k \rightarrow \infty$, and why? (explain using your formula).

(2) You play the following “game,” which involves a decisions made on the hour. If you are exercising this hour, then for sure the next hour you will start studying. If you are studying, then at the end of the hour you toss a fair coin: heads you continue studying, tails you switch to exercise. (This basic model doesn’t include eating or sleeping. Details!)

(a) Let the ordering be $\mathbf{x} = \begin{bmatrix} \text{probability of exercising} \\ \text{probability of studying} \end{bmatrix}$. Write a stochastic matrix A whose Markov chain models the game.

(b) Is the matrix A *regular*? Explain.

(c) Say your initial vector is $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To what, if anything, does your probability of studying tend in the long-time limit $k \rightarrow \infty$? Explain.

(d) Prove the claim from class that any stochastic matrix A must have an eigenvalue 1. [Hint: find a simple eigenvector of A^T .]

- (3) The web consists of three pages. a links to b. b links to c. c links to a and b. Use the PageRank algorithm with $\alpha = 1$.
- (a) Write the stochastic matrix then solve for the vector of importances (order it abc and normalize so that the vector sums to one).

- (b) In a rather selfish effort to increase its importance score, page a removes its link to b. (Make sure you understand how the algorithm deals with such a situation.) What is the new vector of importances? Did the effort work?

BONUS If the PageRank Markov chain iteration (with $\alpha = 1$) was used to approximate solutions to (a) and (b), would they converge, and why?