Your name:

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Math 22 Summer 2017, Homework 6, due Fri August 4 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

(1) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation with standard matrix A.

(a) What are the possible values for the rank of A?
Solution: The rank of A is 0, 1, 2, or 3 since A can have 0, 1, 2, or 3 pivots.
Grading: 1 point for correct ranks.

(b) What are the possible values for the dimension of $\operatorname{Nul} A$?
Solution: By the rank theorem and the previous part of this problem, the dimension of $\operatorname{Nul} A$ is 5, 4, 3, or 2.
Grading: 1 point for correct dimensions.

(c) Suppose now that the T from above is also onto. What are the possible values for the dimension of $\operatorname{Nul} A$?
Solution: For T to be onto, A must have a pivot in every row and therefore must have exactly 3 pivots. But this means A has rank 3 which (by the rank theorem) implies that $\operatorname{Nul} A$ has dimension 2.
Grading: 2 points total. 1 point for correct dimension. 1 point for explanation.

(d) Let $A = \begin{bmatrix} 2 & 0 & -1 & 2 & 3 \\ 4 & 0 & -2 & 4 & 6 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$. Find bases for $\operatorname{Col} A$, $\operatorname{Row} A$, and $\operatorname{Nul} A$ (In this question you do not need to show your working.)

Solution: $\operatorname{Col} A$ has basis $\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$, $\operatorname{Row} A$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1/2 \\ 3/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$,

and $\operatorname{Nul} A$ has basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.
Grading: 3 points total. 1 point for each of the 3 bases.

(2) Let $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(a) A and B have the same characteristic polynomial. Find this polynomial and explain what form and/or features that A and B have in common make the polynomial the same.

Solution: The common characteristic polynomial is $(\lambda + 2)^2(\lambda - 3)^2$. A and B are (upper) triangular matrices. The eigenvalues of a triangular matrix are just the entries along the main diagonal. Since these matrices all have the same diagonal, they all have the same eigenvalues and (algebraic) multiplicities. Since charpolys are determined by eigenvalues and multiplicities, it follows that both charpolys are the same.

Grading: 2 points total. 1 point for the common charpoly. 1 point for explanation.

(b) From the previous part we see that each matrix has exactly 2 eigenvalues (call them λ_1 and λ_2 with $\lambda_1 < \lambda_2$). Compute the dimensions of both eigenspaces for both of the matrices A and B. Use this information to fill in the table below. Be sure to include λ_1 and λ_2 in the table. It is not necessary to include *all* the computations involved, but please describe how you computed the dimensions.

matrix	dimension of $\lambda_1 =$ eigenspace	dimension of $\lambda_2 =$ eigenspace
A		
B		

Solution:

matrix	dimension of $\lambda_1 = -2$ eigenspace	dimension of $\lambda_2 = 3$ eigenspace
A	2	1
B	2	2

In general, given an $n \times n$ matrix A with eigenvalue λ, the dimension of the λ-eigenspace is the dimension of $\operatorname{Nul}(A - \lambda I_n)$ where I_n is the $n \times n$ identity matrix as usual. In particular, using B from above, we see that $B - \lambda_1 I_n$ and $B - \lambda_2 I_n$ both have 2 pivots and therefore correspond to dimesion 2 null spaces. For A (from above), $A - \lambda_1 I_n$ has 2 pivots and thus corresponds to a dimension 2 null space, but $A - \lambda_2 I_n$ has 3 pivots and therefore the $\lambda_2 = 3$ eigenspace of A has dimension 1.

Grading: 3 points total. 1 point for all dimensions of $\lambda = -2$ eigenspace. 1 point for all dimensions of $\lambda = 3$ eigenspace. 1 point for explanation. The explanation just needs to convince the reader of understanding.

(c) Find a basis for the λ_2 eigenspace of B:

Solution:

By row reducing $B - \lambda_2 I_n = B - 3I_n$, we see that $\operatorname{Nul}(B - 3I_n)$ has solutions $x_1 = 0, x_2 = 0$ and x_3, x_4 free. Thus $\operatorname{Nul}(B - 3I_n)$ has basis

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Grading: 1 point for basis. No explanation necessary.

(3) Let $A = \begin{bmatrix} 7 & h \\ 2 & 11 \end{bmatrix}$.

(a) For what value(s) of h does A have an eigenvalue of (algebraic) multiplicity two?
Solution: $\det(A - \lambda I_2) = (7-\lambda)(11-\lambda) - 2h = \lambda^2 - (7+11) + (7+11-2h)$ is the charpoly of A in terms of h. Having a (algebraic) multiplicity two eigenvalue means λ is a multiplicity two zero of this quadratic polynomial. Recall (from the quadratic equation) that $ax^2 + bx + c$ has a root of multiplicity two precisely when $\Delta = 0$ where $\Delta = b^2 - 4ac$. In our case, we have $\Delta = (-(7+11))^2 - 4 \cdot 1 \cdot (7+11-2h)$ which is equal to $16 + 8h$. Thus, $\Delta = 0$ precisely when $h = -2$.
Grading: 2 points total. 1 point for $h = -2$. 1 point for explanation.

(b) Let λ be the eigenvalue of (algebraic) multiplicity two from the previous part. Compute a basis for the λ eigenspace of A.
Solution: Using $h = -2$, we get that the charpoly of A is $(\lambda - 9)^2$, so that $\lambda = 9$ is the (algebraic) multiplicity two eigenvalue under consideration. To compute (a basis for) the $\lambda = 9$ eigenspace of A we compute the null space of

$$A - 9I_2 = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

to get

$$\text{basis for } \lambda = 9 \text{ eigenspace} = \text{basis for } \operatorname{Nul}(A - 9I_2) = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Grading: 2 points total. 1 point for a basis. 1 point for explanation.

(c) Let $P = \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}$, and let $B = P^{-1}AP$ (with A defined by h from part (a)). Compute B and prove that B and A have the same eigenvalues.
Solution: First note that

$$\underbrace{\begin{bmatrix} 9 & 1 \\ 0 & 9 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 0 & 1/2 \\ 1 & 1 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 7 & -2 \\ 2 & 11 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}}_{P}.$$

At this point you can just prove directly that the (triangular) matrix B has the same charpoly as A (already computed above). You can also reproduce the proof of Theorem 4 in section 5.2.
Grading: 3 points total. 1 point for computing B. 2 points for a proof that A and B have the same eigenvalues.