Barnett
7/28/17

SOLUTIONS

Your name:

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Math 22 Summer 2017, Homework 5, due Fri July 28 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

8 points

(1) Let $A = \begin{bmatrix} 5 & -1 & -3 & 11 \\ -2 & 1 & 0 & -5 \\ 3 & -2 & 1 & 8 \end{bmatrix}$. $\xrightarrow{\text{row reduce}}$ $\begin{bmatrix} 2 & -1 & 0 & 5 \\ 10 & -2 & -6 & 22 \\ 6 & -4 & 2 & 16 \end{bmatrix}$ scaled so can cancel w/o using fractions...

[3 pts] (a) Find a basis for Col A.

$$\begin{array}{l} r_2 \leftarrow r_2 - 5r_1 \\ r_3 \leftarrow r_3 - 3r_1 \end{array} \sim \begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 3 & -6 & -3 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns are #1, #2 : look up these cols in A.

$$\text{basis} = \left\{ \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}$$

[1 pt] (b) What is the dimension of the subspace spanned by the columns of A? dim = 2

ie, Col A.

[2 pts] (c) Find a basis for Nul A.

REF is $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

free: x_3 x_4

soln. set for homog. lin. sys:

$$\begin{aligned} x_1 &= 1x_3 - 2x_4 \\ x_2 &= 2x_3 + 1x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

$$\text{so } \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_4 \qquad \text{Basis} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

[2 pts] hard (d) Write, and explain, an inequality satisfied by the dimension of the nullspace of *any* matrix B formed by deleting rows from A. [Hint: what is a subspace of what?]

Nul A is a subspace of Nul B, because removing linear equations from the list to be satisfied cannot shrink the soln set.

I.e. $A\vec{x} = \vec{0} \Rightarrow B\vec{x} = \vec{0}$, so $\vec{x} \in \text{Nul}\,A \Rightarrow \vec{x} \in \text{Nul}\,B$.

Apply theorem 11 (Ch. 4): $\dim \text{Nul}\,B \geq \dim \text{Nul}\,A = 2$.

6 pts.

(2) Consider a linear transformation $T : V \to W$ where V is a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and W is a vector space with basis $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$.

[3 pts] (a) Let $\mathbf{x}, \mathbf{y}$ be in V. Prove that $[\mathbf{x} + \mathbf{y}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{B}} + [\mathbf{y}]_{\mathcal{B}}$. (Here you may use results up to and including the Unique Representation Theorem. Keep in mind V can be any vector space, not necessarily $\mathbb{R}^n$.)

[See pf of Thm. 8 on p. 221 of book]

$\vec{x} = c_1\vec{b}_1 + \cdots c_n\vec{b}_n$ where $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\vec{y} = d_1\vec{b}_1 + \cdots d_n\vec{b}_n$ " $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$

so by vector space axioms, $\vec{x} + \vec{y} = (c_1 + d_1)\vec{b}_1 + \cdots (c_n + d_n)\vec{b}_n$

by the unique rep. thm. (Thm 7) we can read off the coefficients as those of $[\vec{x}+\vec{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$.

where vector addition in $\mathbb{R}^n$ is used.

[3pts] hard

(b) Suppose the map T is onto. What relation between n and m must hold? Prove your answer. [Hint: construct a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ using coordinate maps, then prove that map is onto]

Let $S : \mathbb{R}^n \to \mathbb{R}^m$, with $S(\vec{x}) = [T(x_1\vec{b}_1 + \cdots x_n\vec{b}_n)]_{\mathcal{C}}$

ie S is composition of: (inverse coord map), then T, then coord map.

ie
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "{[\vec{v}]_{\mathcal{B}} \to \vec{v}}", "\text{inv. coord map}"'] \arrow[rrr, bend right, "S"'] & V \arrow[r, "T", "\text{given}"'] & W \arrow[r, "{\vec{w} \to [\vec{w}]_{\mathcal{C}}}", "\text{coord map}"'] & \mathbb{R}^m \end{tikzcd}$$

This map S has a standard matrix A, which is $m \times n$.

Since T is onto, and the coord maps are isomorphisms (§4.4), S is also onto. (This could be proved in more detail: let $\vec{y}$ in $\mathbb{R}^m$ be arbitrary, then $\vec{w} = y_1\vec{c}_1 + \cdots y_m\vec{c}_m$ is the image of some $\vec{v}$ in V because T is onto. Finally, $[\vec{v}]_{\mathcal{B}}$ exists because the coord map is a map. So $\vec{y}$ is image of $[\vec{v}]_{\mathcal{B}}$ under S, so S is onto.)

The std matrix of any onto map $\mathbb{R}^n \to \mathbb{R}^m$ must have pivot in every row, so $m \leq n$.

6 pts

(3) [3 pts] (a) Recall the standard basis $\{1, t, t^2\}$ for $\mathbb{P}_2$. When we express these monomials about a new origin $t = a$, we get the set $\{1, (t-a), (t-a)^2\}$. Prove, for any a, that this set is also a basis for $\mathbb{P}_2$. [Hint: you many use that the coordinate map for the standard basis is an isomorphism.]

We use isomorphism to work in $\mathbb{R}^3$: $1 \xrightarrow{\text{std. coord map}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$t-a \longrightarrow \begin{bmatrix} -a \\ 1 \\ 0 \end{bmatrix}$

$(t-a)^2 = t^2 - 2at + a^2$

So $(t-a)^2 \longrightarrow \begin{bmatrix} a^2 \\ -2a \\ 1 \end{bmatrix}$

Stack:

$A = \begin{bmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{bmatrix}$ is already in REF, showing full rank $\Rightarrow$ cols. are basis for $\mathbb{R}^3$.

Thus the set was a basis for $\mathbb{P}_2$, by isomorphism.

[3 pts] (b) The subspace $H = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is in the span of the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Explain whether $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for H.

No: Span$\{b_1, b_2\} = \mathbb{R}^2$ the whole plane,

which is bigger than H.

Proof: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \text{Span}\{b_1, b_2\}$, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin H$.

So, even though $\{b_1, b_2\}$ are Lin. Indep., they are not a basis for H (which happens to be 1-dim).