Your name:

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Math 22 Summer 2017, Homework 4, due Fri July 21 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

(1) (a) Find the determinant of the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 0 & 4 & 7 \\ 1 & 0 & 5 & 4 \\ 17 & 2 & 38 & 99 \end{bmatrix}$

Solution:

$$\det A = 0 + 0 + 0 + (-1)^{4+2} \cdot 2 \cdot \det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 1 & 5 & 7 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 3 & 4 \end{bmatrix}$$

$$= 2(-1) \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} = (2)(-1)(1)(3)(1) = -6.$$

Grading: 3 points total for the correct determinant.

(b) Is A from above invertible, and why?

Solution: Yes. $\det A \neq 0$ if and only if A is invertible.

Grading: 1 point total for correct answer with valid explanation.

(c) Prove that, if A is any $n \times n$ matrix and P is an invertible $n \times n$ matrix, that det $PAP^{-1} =$ det A. [Please state when any properties of det are used.]

Solution: Since the determinant map from matrices to $\mathbb{R}$ is multiplicative we have

$$\det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}).$$

Since multiplication in $\mathbb{R}$ is commutative, we have

$$\det(P)\det(A)\det(P^{-1}) = \det(P)\det(P^{-1})\det(A).$$

Now use multiplicativity again to get that

$$\det(P)\det(P^{-1}) = \det(PP^{-1}) = \det(I_n) = 1.$$

Thus, $\det(PAP^{-1}) = 1 \cdot \det(A) = \det(A)$.

Grading: 2 points total. 1 point for using multiplicativity of det. 1 point for finishing the proof.

(2) Is each of the following sets a vector space? Explain what result(s) you used to prove your claim. You may assume that $\mathbb{R}^n$ is a vector space.

(a) $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad a+b+c=0 \quad \text{and} \quad 3a-b+2c=0 \right\}$

Solution: Yes. Let H be the above subset of $\mathbb{R}^3$. Notice $H = \text{Nul}A$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

and we proved that null spaces are subspaces. Since H is a subspace of a vector space (namely $\mathbb{R}^3$), H is also a vector space. Alternatively, this can be verified by the definition, but it is tedious.

Grading: 2 points total. 1 point for correct answer. 1 point for explanation.

(b) The set of vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, where t takes all real values.

Solution: No. The zero vector (in $\mathbb{R}^2$) does not lie in this set. If $2+t=0$, then $t=-2$ and $1+3t \neq 0$. If $1+3t=0$, then $t=-1/3$ and $2+t \neq 0$.

Grading: 2 points total. 1 point for correct answer. 1 point for explanation.

(c) The set of vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, where t takes all real values.

Solution: Note that

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2t \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (1+2t) \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and as t ranges over $\mathbb{R}$, so does $1+2t$. Thus, the above subset of $\mathbb{R}^2$ is the same as

$$\left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ in } \mathbb{R} \right\}$$

which is the span of a vector (a line) and hence a subspace of $\mathbb{R}^2$. Thus, this set is a vector space since subspaces of vector spaces are vector spaces.

Grading: 2 points total. 1 point for correct answer. 1 point for explanation.

(d) The set of functions of the form $a + bt^2$, where a and b are real. [Hint: relate this set to $\mathbb{P}_2$]

Solution: Recall that $\mathbb{P}_2$ (quadratic polynomials with coefficients in $\mathbb{R}$) is a vector space with (standard) basis $\{1, t, t^2\}$. Let H be the above subset of $\mathbb{P}_2$. We recognize that H is the span of $\{1, t^2\}$ and is therefore a subspace of $\mathbb{P}_2$. H is a subspace of a vector space and hence a vector space.

Grading: 2 points total. 1 point for correct answer. 1 point for explanation.

(3) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 3 & 0 & 6 & -3 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & -1 \end{bmatrix}.$$

(a) Write the kernel of T (null space of A) in form of a span.
Solution: A is row equivalent (with just 2 row operations) to

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The null space of this matrix can be described by the parametric vector form of the solutions to $A\mathbf{x} = \mathbf{0}$ which is given by

$$\left\{ x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_3, x_4 \text{ in } \mathbb{R} \right\}$$

Grading: 2 points total. 1 point for recognizing kernel as solutions to $A\mathbf{x} = \mathbf{0}$. 1 point for computing parametric vector form of solutions.

(b) Is Col A equal to $\mathbb{R}^3$? (explain)
Solution: No. Any $\mathbf{b}$ in $\mathbb{R}^3$ with last coordinate nonzero cannot be in the span of the columns. This can be shown by row reducing an augmented matrix. We can say more now using post hw4 material. Take the dependence relations from the REF of A and use them to eliminate redundant vectors in Col$A$.
Grading: 2 points total. 1 point for correct answer. 1 point for explanation.

(c) True/False: If the kernel of any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a single point, then $n \geq m$? (explain, or, if false, correct the deduction)
Solution: False. Take $\mathbb{R} \to \mathbb{R}^2$ defined by $x \mapsto (x, 0)$. We can correct this by changing the conclusion of this statement to be that T is one-to-one, which also implies that $m \geq n$.
Grading: 1 point for correct answer. 1 point for corrected statement.

BONUS Find a 2×2 matrix A for whom Col A and Nul A are equal (careful: equal as sets).