Your name:

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Math 22 Summer 2017, Homework 3, due Fri July 14 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

(1) **(6 points)** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix}.$$

(a) Find the standard matrix for T.

Answer: The standard matrix of T is given by

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & 2 \end{bmatrix}.$$

Alternative Answer: There was a "typo" in the homework where the map was instead

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_1 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix}.$$

In this case we have

$$A = \begin{bmatrix} -1 & 0 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}.$$

Points for part (a): This part is worth 2 points. 1 point for attempting to compute A via applying T to $\mathbf{e}_1, \mathbf{e}_2$. 1 point for carrying out the computation correctly.

(b) Is T one-to-one? Explain why or why not.

Answer: A is row-equivalent to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

but any echelon form will do. Since A has a pivot in every column, T is one-to-one. Note that the answer for this part is the same regardless of which version of part (a) was completed.

(c) Is T onto? Explain why or why not.

Answer: A is row-equivalent to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

but any echelon form will do. Since A does not have a pivot in every row, T is not onto. Note that the answer for this part is the same regardless of which version of part (a) was completed.

Points for parts (b) and (c): For parts (b) and (c) row reducing A to an echelon form correctly is worth 2 points. Correctly identifying whether the map is one-to-one with explanation is worth 1 point. Correctly identifying whether the map is onto with explanation is worth 1 point.

(2) (**8 points**) Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -4 \\ 1 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

(a) If A is not invertible, prove it, else, if A is invertible, find A^{-1}.
Answer: A is row-equivalent to I_3, and the row operations transforming A to I_3 transform I_3 to A^{-1}:

$$\underbrace{\left[\begin{array}{cccccc} -2 & 1 & -4 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]}_{[A|I_3]} \sim \left[\begin{array}{cccccc} 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right] \sim \underbrace{\left[\begin{array}{cccccc} 1 & 0 & 0 & -4 & -7 & -2 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right]}_{[I_3|A^{-1}]}.$$

Points for part (a): 2 points for attempting to row reduce $[A|I_3]$. 2 points for correctly row reducing to get $[I_3|A^{-1}]$.

(b) Use the previous part of this problem to express the vector $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ as a linear combination of the columns of A.
Answer: The weights for the desired linear combination are given by the (unique since A is invertible) solution to the matrix equation

$$A\mathbf{x} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$

But multiplying on the left by A^{-1} we see that

$$\mathbf{x} = A^{-1}\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 & -7 & -2 \\ 1 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ 4 \\ 9 \end{bmatrix}.$$

This tells us that $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = -15\mathbf{a}_1 + 4\mathbf{a}_2 + 9\mathbf{a}_3$.

Points for part (b): 1 point for setting up the correct matrix equation. 1 point for solving using A^{-1}.

(c) Let A be an arbitrary 4×4 invertible matrix. Write a linear system involving the matrix A whose solution gives the third column of A^{-1}. (Write your linear system as a matrix equation).
Answer: A matrix equation corresponding to a linear system has the form $A\mathbf{x} =$

b. For a given **b**, this system has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. Let $A^{-1} = \begin{bmatrix} \mathfrak{a}_1 & \mathfrak{a}_2 & \mathfrak{a}_3 & \mathfrak{a}_4 \end{bmatrix}$. We are trying to choose **b** so that $A^{-1}\mathbf{b} = \mathfrak{a}_3$. But,

$$A^{-1}\mathbf{b} = b_1\mathfrak{a}_1 + b_2\mathfrak{a}_2 + b_3\mathfrak{a}_3 + b_4\mathfrak{a}_4$$

so we see that setting $b_1 = b_2 = b_4 = 0$ and $b_3 = 1$ works. In other words, the system $A\mathbf{x} = \mathbf{e}_3$ with $\mathbf{e}_3$ in $\mathbb{R}^4$ is the desired linear system with solution $\mathfrak{a}_3$.

Points for part (c): 1 point for evidence of understanding what the question is asking (i.e. that we are looking for the solution of the matrix equation $A\mathbf{x} = \mathbf{b}$ to have some property). 1 point for being able to set up the correct system.

(3) (**6 points**) A certain disease affects the human population as follows over the course of one year: Two thirds of the initially healthy (H) humans become sick (S), while the rest remain healthy. Over that same year, one third of the initially sick recover and become healthy, one third remains sick, and one third sadly dies.

(a) Stacking the H and S populations into a column vector **x**, write the population vector at year $k+1$ in terms of that at year k. Be sure to give the matrix. [Note: the dead are not included in the column vector nor the matrix.]

Answer: For k a nonnegative integer, let h_k be the healthy population at k years, let s_k be the sick population at k years, and let $\mathbf{x}_k = \begin{bmatrix} h_k \\ s_k \end{bmatrix}$. Then this system is described by the recurrence relation

$$A\mathbf{x}_k = \mathbf{x}_{k+1}$$

where

$$A = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}.$$

Then

$$\mathbf{x}_{k+1} = \begin{bmatrix} h_{k+1} \\ s_{k+1} \end{bmatrix} = A\mathbf{x}_k = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} h_k \\ s_k \end{bmatrix} = \begin{bmatrix} (1/3)h_k + (1/3)s_k \\ (2/3)h_k + (1/3)s_k \end{bmatrix}.$$

Points for part (a): 1 point for getting the correct matrix A. 1 point for using A to write the correct population vector.

(b) If the populations at "year zero" are 900 healthy and zero sick, what are the populations at year 2? (ie, after two years of dynamics).

Answer: Using the notation from part (a), we are given $\mathbf{x}_0 = \begin{bmatrix} 900 \\ 0 \end{bmatrix}$ and we are looking for $\mathbf{x}_2$. We get

$$\mathbf{x}_2 = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \left(\begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 900 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 300 \\ 600 \end{bmatrix} = \begin{bmatrix} 300 \\ 400 \end{bmatrix}.$$

So there are 300 healthy people and 400 sick people at year 2.

Points for part (b): 1 point for writing the correct initial vector. 1 point for correctly computing $\mathbf{x}_2$.

(c) Find a *matrix* that takes any population vector at year k to that at year $k+2$. [Hint: check against part b.]

Answer: Incrementing the system by 2 years is the same as multiplying by A twice. That is, by multiplying by A^2. We get

$$A^2 = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/9 \\ 4/9 & 1/3 \end{bmatrix}$$

as the desired matrix, and you can check that $A^2\mathbf{x}_0 = \mathbf{x}_2$ as it must.

Points for part (c): 1 point for understanding that such a matrix M must satisfy $M\mathbf{x}_0 = \mathbf{x}_2$. 1 point for computing A^2 correctly.