7/7/17.
Barnett

~ SOLUTIONS ~

Your name:

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Math 22 Summer 2017, Homework 2, due Fri July 7 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

6 pts. (1) Let h be a scalar, and consider the set of three vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -6 \\ 3 \\ 9 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -3 \\ 2(h-1) \\ 7 \end{bmatrix}.$$

4. (a) With $h = 1/2$, is the set linearly independent, and why? If not, give a dependence relation between $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Stack as matrix : $\begin{bmatrix} 3 & -6 & -3 \\ -2 & 3 & -1 \\ -4 & 9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{bmatrix}$

REF

$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (2 pts) x_3 free

$\Rightarrow$ The set is not L.I.

$\left.\begin{aligned} x_1 &= -5x_3 \\ x_2 &= -3x_3 \\ x_3 &= x_3 \end{aligned}\right\} \quad \vec{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \in$ Solution set for $A\vec{x} = \vec{0}$

$\Rightarrow$ Dependence rel. is $-5\vec{v}_1 - 3\vec{v}_2 + \vec{v}_3 = \vec{0}$ (2 pts for this)

2. (b) With $h = 0$, is the set linearly independent, and why? If not, give a dependence relation between $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$\begin{bmatrix} 3 & -6 & -3 \\ -2 & 3 & -2 \\ -4 & 9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ & -1 & -4 \\ & 1 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & -1 \\ & 1 & 4 \\ & & 1 \end{bmatrix}$ EF

pivot in every column

$\Rightarrow$ no free vars.

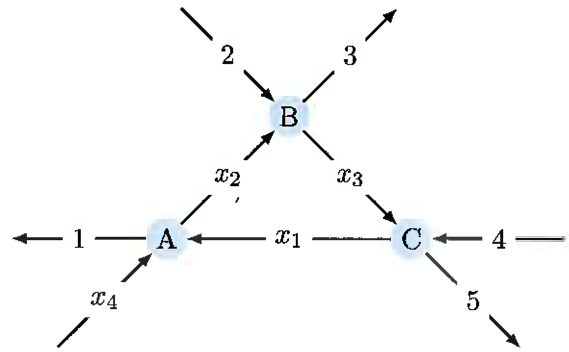
$\Rightarrow A\vec{x} = \vec{0}$ has only trivial soln.

$\Rightarrow$ set is L.I.

[1 pt for why].

7pts.

(2) Consider the following network flow diagram where the numbers indicate known flows, and x_1, x_2, x_3, x_4 indicate unknown flows on their respective edges.



4.

(a) Find the *reduced* echelon form for the corresponding linear system.

node	in		out
A:	$x_1 + x_4$	$=$	$1 + x_2$
B:	$2 + x_2$	$=$	$3 + x_3$
C:	$4 + x_3$	$=$	$5 + x_1$ ← [2pts]

stack & simplify, aug matrix of lin. sys:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R_3 \leftarrow R_3 + R_1} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \text{EF} \underset{\sim}{R_1 \leftarrow R_1 + R_2 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

↑ x_3 free.

2.

(b) If a solution exists, write the *parametric vector form* of the solution set.

$x_1 = -1 + x_3$

$x_2 = 1 + x_3$

$x_3 = x_3$

$x_4 = 3$

$$\bar{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} t, \quad t \in \mathbb{R}.$$

1.

(c) Assume all flows are nonnegative. What constraint does this impose on x_3?

$x_3 \geq 1$, otherwise x_1 goes negative.

7pts

(3) (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

3.

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3, \quad \text{with}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}, \text{ and } \quad \mathbf{v}_3 = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}.$$

Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$. Explain why your answer is correct.

We don't need to use the theorem from §1.9.

Since $T(\vec{x})$ is a linear combination of $\vec{v}_1, \cdots \vec{v}_3$ using the weights $x_1, \cdots x_3$, this is the <u>definition</u> of matrix-vector multiplication $A\vec{x}$, for $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 3 & 4 & 7 \\ -2 & -3 & 1 \\ 5 & 1 & 2 \end{bmatrix}$

It is also acceptable to show that $T(\vec{x})$ & $A\vec{x}$ <u>both</u> have 1st entry $3x_1 + 4x_2 + 7x_3$, 2nd entry $-2x_1 - \cdots$ etc.

4.

(b) Let $\mathbf{p}$ be any vector in $\mathbb{R}^n$ and let $\mathbf{v}$ be any nonzero vector in $\mathbb{R}^n$. Consider the line ℓ defined by the parametric equation

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}, \quad t \text{ in } \mathbb{R}.$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Show that the image of ℓ under the map T is either a line in $\mathbb{R}^n$ or a single point.

You are given T, $\vec{p}$, $\vec{v}$.

Use definition of T being linear:

Then $T(\vec{x}) = T(\vec{p} + t\vec{v}) = T(\vec{p}) + T(t\vec{v}) = T(\vec{p}) + tT(\vec{v})$

[2 pts for exploiting linearity]

rule i) for T being linear

rule ii) for T being linear.

Two cases:

- $T(\vec{v}) = \vec{0}$: then $T(\vec{x}) = T(\vec{p}) + 0 = T(\vec{p})$

  We now consider varying t, and see that $T(\vec{x})$ does <u>not</u> vary. Image of ℓ under T is $\{T(\vec{p})\}$, a single point.

[2 pts for realizing the two cases]

- $T(\vec{v}) \neq \vec{0}$: $T(\vec{x}) = \vec{q} + t\vec{u}$ for some $\vec{q}$, & $\vec{u} \neq \vec{0}$

  This, as t varies, is the parametric form of a line.