6/30/17
Barnett

Your name: ~ SOLUTIONS ~

Instructor (please circle): Alex Barnett Michael Musty

Math 22 Summer 2017, Homework 1, due Fri June 30 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

(7 pts) (1) Given the system of equations, answer the following questions.

$$\begin{aligned} 2x_1 + 4x_3 &= 6 \\ -2x_1 + 3x_2 + 11x_3 &= 15 \\ -4x_1 + 3x_2 + 7x_3 &= 9 \end{aligned}$$

5. (a) Write the augmented matrix and transform it to reduced echelon form:

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 6 \\ -2 & 3 & 11 & 15 \\ -4 & 3 & 7 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 3 & 15 & 21 \\ 0 & 3 & 15 & 21 \end{array}\right]$$

$R_1 \leftarrow \frac{1}{2}R_1$ then
$R_2 \leftarrow R_2 + 2R_1$
$R_3 \leftarrow R_3 + 4R_1$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

it's in EF & also REF.

helpful but not crucial.

working must be shown (intermediate steps)

x_3 free

−1 for copying or small arith. error.

pivots & free var don't have to be stated, but helps understand (b).

2. (b) Write the *general* solution to the linear system, if there is one:

$$\begin{aligned} x_1 &= 3 - 2x_3 \\ x_2 &= 7 - 5x_3 \\ x_3 &= x_3 \end{aligned}$$

(−1) if forget x_3 is part of general soln.

(2) True or false (no working needed, just circle the answer):

pivot

(a) T / F: The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ is in echelon form.

would have to be zero since pivot above.

(b) T / F: If an echelon form of an augmented matrix has a row of all zeros (including the right-hand side), then the linear system must be consistent.

there still could be a row $[0 \cdots 0 \mid b]$ which makes inconsistent. $b \neq 0$

(c) T / F: Two linear systems that are row equivalent always reduce to the same reduced echelon form.

Doing row ops. doesn't change the REF you'll get to (which is unique, given A)

(d) T / F: Given any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$, it always holds that $\underbrace{\text{Span}\{\mathbf{v}, \mathbf{v}, \mathbf{u}\}}_{=:T} = \underbrace{\text{Span}\{\mathbf{u}, \mathbf{v}\}}_{=:S}$.

can check $S \subseteq T$ & $T \subseteq S$

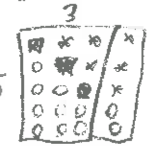
Duplicate & reordering don't affect the Span

(e) T / F: A linear system with more equations than unknowns cannot have a unique solution.

Eg 5×3 w/ 3 pivots

3

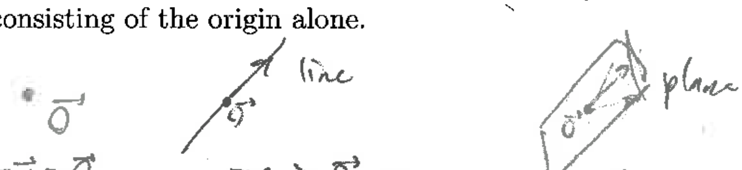
5



no free vars $\Rightarrow$ unique

aug matrix in EF: cons. (for this RHS) unique.

(f) T / F: The span of two vectors in $\mathbb{R}^m$ is either a plane passing through the origin, a line passing through the origin, or the set consisting of the origin alone.



case: $\vec{u} = \vec{v} = \vec{0}$ | one is $\vec{0}$ or they're parallel. | otherwise

BONUS: Suppose two 1×2 linear systems have the same *solution sets*. Either prove that their augmented matrices have exactly the same reduced echelon form, or find a counterexample. [This is a challenge problem inspired by a student question; you may need more space!]

True — see last page.

up to +2 avail.

[7 tot] (3) For what value(s) of the real number h can the vector $\begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$ be written as a linear combination of the three vectors $\underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$, $\underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ h \end{bmatrix}}$, and $\underset{\vec{v}_3}{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}$?

4.

It doesn't matter what order the three are in, so make $\vec{v}_2$ the last one: Need the following lin. sys. aug. matrix to be consistent:

Ok to do in original order too.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & h & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & h & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & h-3 & 0 \end{array}\right]$$

2 pts.

two cases: $h=3$: no pivot in last row, but $[0\ 0\ 0\ |\ 0]$, so, consistent

interpret, 2 pts.

OR $h \neq 3$: $[0\ 0\ \blacksquare\ |\ 0]$ also consistent.

$\Rightarrow$ for all h real.

3.

For what value(s) of h is the span of the set of three vectors equal to $\mathbb{R}^3$?

equivalently: there's pivot in every row of coeff matrix.

Ignoring RHS above, EF is same:

$$\text{coeff matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & h-3 \end{bmatrix}$$

? is a pivot?

$h=3$: not a pivot in last row

$h \neq 3$: pivots in every row

Ans: $h \neq 3$

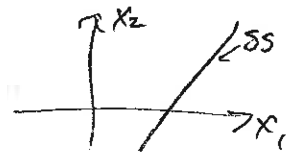
-1 if correct REF & ans. but no ~~[illegible]~~ explanation.

BONUS : There are 3 types of soln. set :

i) $SS = \{\}$ empty. Then REF of aug. matrix must be $[0\ 0 \mid 1]$

ii) $SS = \mathbb{R}^2$ Then REF must be $[0\ 0 \mid 0]$

iii) SS = a line.



Suppose line is horizontal, then x_2 is const while x_1 varies.

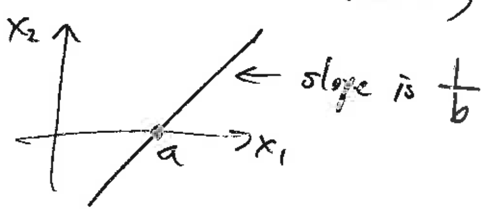
Thus REF must be $[0\ 1 \mid a]$
x_1 free

The only way two such SS's can be equal (as sets) is if their REFs have same a.

$\Rightarrow$ REF uniquely determined.

Otherwise, REF must be $[1\ b \mid a]$ for some $a, b \in \mathbb{R}$

Let's prove geometrically that if two REFs $[1\ b \mid a]$ and $[1\ d \mid c]$ have the same non-horizontal line SS, then $b = d$ & $a = c$, so they are the same REF:



Since such a line hits the x_1 axis at a, and has slope $1/b$, these two numbers are uniquely determined.

(You can also check by substituting two values $x_2 = 0, 1$ & requiring x_1 be the same for the two REFs)

Phew! I wonder what happens for general $m \times n$ ...?