Math 22 Fall 2013 Midterm Exam II
Tuesday, October 29, 2013

PRINT NAME: Solutions

INSTRUCTIONS: READ CAREFULLY!

This is a closed book, closed notes exam. Use of calculators is not permitted. You have two hours, do all problems.

On all **free response** questions below you must show your step-by-step work and make sure it is clear *how* you arrived at your solution. Whenever you answer a question, don't just say "Yes" or "No", but always justify your answers. No credit is given for solutions without appropriate work or justification. You will receive partial credit for partially correct answers.

For multiple choice and True/False questions, no justification is necessary. Therefore, **leave no multiple choice question unanswered!** Guessing is allowed: A wrong guess does not cost you more points than leaving the question unanswered.

The Honor Principle requires that you neither give nor receive any aid on this exam.

GOOD LUCK!

FERPA waiver: By my signature I relinquish my FERPA rights in the following context: This exam paper may be returned en masse with others in the class and I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructors office to retrieve my examination paper.

FERPA waiver signature:

1. Let A be the 3×3 matrix

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) [3 pt] Calculate $A^2 + 2A + I$.

$$A^2 = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 19 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^2 + 2A + I = \begin{pmatrix} 1 & 6 & 19 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & 12 & 29 \\ 0 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix}$$

or

$$A^2 + 2A + I = (A+I)^2 = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 12 & 29 \\ 0 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix}$$

(b) [3 pt] Assume that A, B are 3×3 matrices. What is wrong with the following "proof" that $(A+B)^2 = A^2 + 2AB + B^2$?

Possible bonus Question

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B) \\ &= A(A+B) + B(A+B) \\ &= A^2 + AB + AB + B^2 \\ &= A^2 + 2AB + B^2\end{aligned}$$

$BA \neq AB$ in general. It's true in (a) b/c $AI = IA$

2. [3 pt] Suppose A, B, C are $n \times n$ invertible matrices. Show that the product ABC is invertible by giving a formula for a matrix D such that $(ABC)D = I$ and $D(ABC) = I$.

$D = C^{-1}B^{-1}A^{-1}$. No further justification needed.

3. [5 pt] Find the inverse matrix of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$(A)\begin{pmatrix} 0 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$ $(B)\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix}$ $(C)\begin{pmatrix} 0 & -1 & 2 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

$(D)\begin{pmatrix} 0 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix}$ $(E)\begin{pmatrix} 0 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ $(F)\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

ANSWER

$\boxed{E}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

Check

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. [5 pt] Determine if the collection of vectors

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

is a basis for $\mathbb{R}^3$. Justify your answer.

$$\begin{pmatrix} 4 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 4 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & \boxed{-7} & 0 \end{pmatrix}$$

Pivot in every column $\Rightarrow$ LI $\Rightarrow$ basis for $\mathbb{R}^3$.

5. [8 pt] Find a basis for Col A and Nul A for the matrix

$$A = \begin{pmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{pmatrix}$$

$$A \sim \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -3 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 2 & 0 & -6 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis for Col(A) : $\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ 7 \\ 6 \end{pmatrix}$

Basis for Nul(A) : $\begin{pmatrix} -2x_2 + 6x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + x_4 \underbrace{\begin{pmatrix} 6 \\ 0 \\ -2 \\ 1 \end{pmatrix}}_{v_2}$

The basis is given by v_1, v_2.

6. Let H be the subspace of $\mathbb{R}^4$ that is the span of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 5 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 4 \\ 6 \\ 0 \\ 0 \end{pmatrix}$$

(a) [5 pt] Find a basis for H.

Note: This is equivalent to finding a basis for $Col(A)$, where $A = (v_1\, v_2\, v_3\, v_4\, v_5)$

$$\begin{pmatrix} 1 & -2 & 3 & 1 & 4 \\ 1 & 0 & 5 & 3 & 6 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & -2 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 1 & 4 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & -2 & -1 & 1 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & -2 & 3 & 1 & 4 \\ 0 & \boxed{1} & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} \end{pmatrix}$$

Pivots in columns $1, 2, 3, 5 \Rightarrow$ a basis is $\{v_1, v_2, v_3, v_5\}$

(b) [2 pt] Is H all of $\mathbb{R}^4$? Explain your answer.

Yes, it's a 4-dim subspace of $\mathbb{R}^4 \Rightarrow$ it's all of $\mathbb{R}^4$

7. [4 pt] If A is a 3×4 matrix of rank 1, does the equation $A\mathbf{x} = 0$ have any non-trivial solutions? Explain your answer.

$$\underbrace{\operatorname{rank}(A)}_{\substack{\shortparallel \\ \#\text{ of pivot columns}}} + \underbrace{\dim(Nul(A))}_{\substack{\shortparallel \\ \#\text{ of free variables of } Ax=0}} = \#\text{ columns} = 4$$

Because rank$(A) = 1$ $Ax=0$ has 3 free variables $\Rightarrow$ the equation $Ax=0$ has a nontrivial solution.

8. [5 pt] What is the determinant of the matrix

$$\begin{array}{c} \downarrow \\ \begin{pmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{pmatrix} \end{array}$$

(A) 0 (B) 2 (C) -4 (D) 6 (E) -8 (F) 10

ANSWER

$\boxed{A}$

$$\det = +1 \det\begin{pmatrix} 1 & 3 & -3 \\ -3 & 4 & 8 \\ 3 & -4 & 4 \end{pmatrix} - 2 \det\begin{pmatrix} -3 & -2 & -4 \\ 1 & 3 & -3 \\ 3 & -4 & 4 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 3 & -3 \\ -3 & 4 & 8 \\ 3 & -4 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 & -3 \\ -3 & 4 & 8 \\ 0 & 0 & 12 \end{pmatrix} = 12 \det\begin{pmatrix} 1 & 3 \\ -3 & 4 \end{pmatrix} = \underset{\substack{\shortparallel \\ 156}}{12 \cdot (13)}$$

$$\det\begin{pmatrix} -3 & -2 & -4 \\ 1 & 3 & -3 \\ 3 & -4 & 4 \end{pmatrix} = (-1)\det\begin{pmatrix} -2 & -4 \\ -4 & 4 \end{pmatrix} + 3\det\begin{pmatrix} -3 & -4 \\ 3 & 4 \end{pmatrix} - (-3)\det\begin{pmatrix} -3 & -2 \\ 3 & -4 \end{pmatrix}$$

$$= (-1)(-24) + 3(0) + 3(18) = 24 + 54 = 78$$

$$\det = 156 - 2(78) = 0$$

Also note (Column 2) $-$ 2 (Column 3) = (Column 4)

9. [4 pt] Suppose A is an $n \times n$ matrix for which $AA^T = I$. Use the properties of the determinant to find the value of $\det(A)$. Explain your answer.

Possible bonus question

$$\underbrace{\det(AA^T)}_{=\det(I)=1} = \det(A)\det(A^T) = \det(A)\det(A) = \det(A)^2$$

$$\det(I) = 1 \implies \det(A) = \pm 1$$

10. [5 pt] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with standard matrix

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 4 & 1 \end{pmatrix}$$

If S is a cube in $\mathbb{R}^3$ whose sides have length 1, and S' is the image of the cube S under the linear transformation T, what is the volume of S'?

$$\text{Volume}(S') = |\det(A)| \cdot \underbrace{\text{Volume}(S)}_{=1} = 13 \cdot 1 = 13$$

$$\det(A) = -(-1)\det\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} + 1\det\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = (1)(10) + (1)(3) = 13$$

11. (a) [5 pt] Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 4 & 0 & -9 \\ -1 & 3 & 5 \\ 4 & 0 & -8 \end{pmatrix}$$

(b) [3 pt] Is the matrix A diagonalizable? Show your work, and explain your conclusion. You do not need to find the explicit matrices D, P.

(a) $\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & 0 & -9 \\ -1 & 3-\lambda & 5 \\ 4 & 0 & -8-\lambda \end{pmatrix} = (3-\lambda)\det\begin{pmatrix} 4-\lambda & -9 \\ 4 & -8-\lambda \end{pmatrix}$

$= (3-\lambda)\left[(4-\lambda)(-8-\lambda)+36\right] = (3-\lambda)\left[\lambda^2+4\lambda+4\right] =$
$= (3-\lambda)(\lambda+2)^2$

$\Rightarrow \lambda_1 = 3, \lambda_2 = -2$ (w/ multiplicity 2)

(b) We need to check that the -2-eigenspace has 2 basis vectors

$$A + 2I = \begin{pmatrix} 6 & 0 & -9 \\ -1 & 5 & 5 \\ 4 & 0 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & -5 \\ 0 & 20 & 14 \\ 0 & 30 & 21 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & -5 & -5 \\ 0 & \boxed{1} & 7/10 \\ 0 & 0 & 0 \end{pmatrix}$$

because there are two pivots the -2-eigenspace is 1-dimensional, so A is not diagonalizable.

12. [5 pt] The matrix A below is diagonalizable,

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 5$.

Diagonalize the matrix A, i.e., find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. You do not have to compute P^{-1}.

$$A - 2I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$A - 5I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

TRUE or FALSE?

For each of the statements below indicate whether it is true or false. You do not need to justify your answers. You lose 2 points out of 10 for each answer that is not correct (or that is left blank). You cannot lose more than 10 points.

13. [10 pt]

(a) True / **False** A square matrix that has two equal rows cannot be invertible.
One row has no pivot. Or $\det(A)=0$ if two rows are equal

(b) True / **False** If H is a 4 dimensional subspace of $\mathbb{R}^n$, and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent vectors in H, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for H.

(c) True / **False** If a 4×7 matrix has 3 pivot columns then the dimension of $\operatorname{Nul} A$ is 1.
$\dim(\operatorname{Nul}(A)) + \operatorname{rank}(A) = 7$
\# of pivots

(d) True / **False** If a 4×7 matrix has rank 4, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$ in $\mathbb{R}^4$.
$\operatorname{Col}(A)$ is a 4-dim subspace of $\mathbb{R}^4 \Rightarrow \operatorname{Col}(A) = \mathbb{R}^4$
$\Rightarrow Ax=b$ is always consistent

(e) True / **False** A linearly independent set of vectors in a subspace H of $\mathbb{R}^n$ is automatically a basis for H.
They must span as well

(f) True / **False** If two $n \times n$ matrices have the same eigenvalues, then they must have the same characteristic polynomial.
Different polynomials can have the same roots.

(g) True / **False** Matrices that are similar have the same determinant.
See midterm 2, fall 2012, 8 (a)

(h) True / **False** If A and B are similar $n \times n$ matrices, and A is invertible, then so is B.
$\det(A) \neq 0 \Rightarrow \det(B) \neq 0$

(i) True / **False** If A is an $n \times n$ matrix, and B, C are $n \times m$ matrices such that $AB = AC$, then it follows that $B = C$.
only true if A is invertible

(j) True / **False** If 5 is an eigenvalue of an invertible matrix A, then $\frac{1}{5}$ must be an eigenvalue of A^{-1}.
$Ax = 5x \Rightarrow x = A^{-1}(5x) = 5(A^{-1}x) \Rightarrow A^{-1}x = \frac{1}{5}x$