Math 22, Summer 2013, Midterm I

Name (Print): *Solutions*

Last	First

Instructions: You are not allowed to use calculators, books, or notes of any kind. You may not look at a classmate's exam for "inspiration." <u>You must explain your reasoning behind each solution to receive full credit</u>. Credit will not be awarded for correct answers with no explanation (with the exception of problem #1).

You may use pages 11-14 of the exam as scratch paper, but any work you intend to be graded should be on the exam itself in the space provided. If you run out of room, clearly indicate which page of scratch paper your solution is on and circle the solution that should be graded.

Before beginning the exam, skim through the problems to verify that you have one true/false question, four free-response questions, and one bonus question.

1. Determine whether each statement below is true or false and indicate your answer by circling the appropriate choice (1pt each):

(a) (True / **False**) If A is a 3×3 matrix with 2 pivot columns, then $Ax = b$ has a solution for each b in $\mathbb{R}^3$.

You're not guaranteed consistency.

(b) (**True** / False) Suppose v_1, v_2, v_3 are vectors in $\mathbb{R}^n$, and b is also a vector in $\mathbb{R}^n$. Then b is in $Span\{v_1, v_2, v_3\}$ if and only if $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$ has a solution.

Definition.

(c) (**True** / False) The columns of a 2×3 matrix A must be linearly dependent.

At most two pivots, need one per column to get LI

NOT A THEOREM → (d) (True / **False**) The matrix $A = \begin{bmatrix} 4 & -h \\ h & 1 \end{bmatrix}$ is not invertible for some scalar h.

QUESTION

$det(A) = 4 + h^2$ which is never zero if $h \in \mathbb{R}$.

(e) (**True** / False) If T is a linear transformation, then T either maps a line with parametric form $x = p + tv$, $t \in \mathbb{R}$, to a line or a point in its codomain.

NOT AN EXAM QUESTION

$T(x) = T(p + tv) = T(p) + tT(v)$. If $T(v) \neq 0$ this is a line, otherwise it's a point.

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2. Let $A = \begin{bmatrix} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

(a) Let $b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Determine whether $Ax = b$ is consistent or inconsistent. If $Ax = b$ is consistent, describe its solution set in <u>parametric vector form</u>. If the system is inconsistent, explain why. (5pts)

$$\begin{pmatrix} 1 & 3 & 5 & 9 & 1 \\ 2 & 4 & 6 & 7 & 2 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 9 & 1 \\ 0 & -2 & -4 & -11 & 0 \\ 0 & -1 & -2 & -5 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 9 & 1 \\ 0 & 1 & 2 & 5 & 1 \\ 0 & -2 & -4 & -11 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 9 & 1 \\ 0 & 1 & 2 & 5 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

echelon form ✓ The system is consistent!

$$\sim \begin{pmatrix} 1 & 3 & 5 & 0 & 19 \\ 0 & 1 & 2 & 0 & 11 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & -14 \\ 0 & 1 & 2 & 0 & 11 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$\begin{cases} x_1 = x_3 - 14 \\ x_2 = -2x_3 + 11 \\ x_4 = -2 \\ x_3 \text{ is free} \end{cases}$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -14 \\ 11 \\ 0 \\ -2 \end{pmatrix}$$

(b) Is $b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ in the span of the columns of A? Explain. (2pts)

Yes, b/c $(A \ b)$ is consistent.

(c) How many columns of A are pivot columns? (3pts)

Three (columns 1, 2, 4)

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(d) Is $Ax = b$ consistent for <u>every</u> b in $\mathbb{R}^3$? Explain. (3pts)

Yes, b/c there is a pivot position in every row.

3. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) Determine the solution set in <u>parametric vector form</u> of the homogeneous equation $A\mathbf{x} = 0$. (2pts)

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \qquad \begin{cases} x_1 = -x_2 \\ x_3 = 0 \\ x_2 \text{ is free} \end{cases} \Rightarrow \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

(b) The aim of the following statement is to provide a <u>geometric</u> description of the solution set found in part (a). Fill in the blanks to make the following statement true (2pts):

"The solution set of $A\mathbf{x} = 0$ obtained in part (a) is a ___line___ in $\mathbb{R}^3$ through ___$(-1, 1, 0)$___ ."

b/c the solution is Span $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(c) Are the columns of A linearly independent? Explain. (2pts)

No, b/c the homogeneous system has nontrivial solutions

(d) Let T be the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^3$, where A is the 3×3 matrix given at the beginning of the problem. Provide an explanation when answering the following 3 questions (2 pts each):

i. Is T linear?

Yes, b/c it's multiplication by A

ii. Is T onto $\mathbb{R}^3$?

No, b/c there is a row with no pivot position

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iii. Is T one-to-one?

No, b/c the columns are not linearly independent (or, there is a column without pivot)

(e) Let $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Is $Ax = b$ consistent? Explain. (2pts)

$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow$

No, it's inconsistent b/c the last row gives the equation $0x_1 + 0x_2 + 0x_3 = 1$, which has no solution

(or, b/c the last column of the augmented matrix is a pivot column)

(f) Let $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Describe geometrically the solution set of $Ax = b$ in relation to the solution set of the homogeneous equation $Ax = 0$. (2pts)

$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $X = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \\ 0 \end{pmatrix}$

The solution set of $Ax = b$ is the solution set of $Ax = 0$ translated to go through the point $\begin{pmatrix} 1/3 \\ 1/3 \\ 0 \end{pmatrix}$ instead of the origin.

Neither part is a midterm I question.

4. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Compute the determinant of A. (3pts)

Using the first column

$$\det(A) = (1)\left[(1)(0)-(1)(1)\right] + (1)\left[(0)(1)-(1)(1)\right] = -2$$

(b) Use elementary row operations to compute the inverse of A. (5pts)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$A^{-1} = \left(\begin{array}{ccc} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

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5. Determine the standard matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first maps the unit vector e_1 to itself and maps the unit vector e_2 to the vector $2e_1 + e_2$, followed by a reflection across the line $x_2 = x_1$. Explain your work to receive full credit. (8 pts)

Not really an exam question, at least in the way it's phrased. If something like this happened, I'd tell you explicitly ⟵ what the reflection does

The reflection maps e_1 to e_2 and e_2 to e_1. Overall,

$$T : \quad e_2 \longmapsto 2e_1 + e_2 \longmapsto 2e_2 + e_1$$
$$e_1 \longmapsto e_1 \longmapsto e_2$$

$$T(e_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad T(e_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$T(x) = Ax \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

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6. **BONUS PROBLEM** Suppose $\{v_1, v_2\}$ is a linearly independent set in $\mathbb{R}^n$. Prove that $\{v_1, v_1 + v_2\}$ is also linearly independent.

This really is a proofwriting problem, so definitely not an exam question.

Proof: Suppose $\{v_1, v_1 + v_2\}$ were linearly dependent. Then we can find $c, d \in \mathbb{R}$ such that

$$c v_1 + d(v_1 + v_2) = 0.$$

But this means that

$$c v_1 + d v_1 + d v_2 = 0 \implies (c+d) v_1 + d v_2 = 0$$

which is a dependence relation for v_1, v_2.

Since v_1, v_2 are LI this is a contradiction. ⨎

Hence $\{v_1, v_1 + v_2\}$ is LI