Math 22: Final Exam

November 16, 2012, 3pm-6pm

Your name (please print): Solutions

Instructions: This is a closed book, closed notes exam. **Use of calculators is not permitted**. Unless otherwise stated, you must justify all of your answers to receive credit - please write in complete sentences in a paragraph structure. You may not give or receive any help on this exam and all questions should be directed to Professor Pauls.

You have **3 hours** to work on all **9** problems. Please do all your work in this exam booklet.

The Honor Principle requires that you neither give nor receive any aid on this exam.

(1) (10 points) Complete the following definitions - remember, state definitions of the terms, not properties of the terms. To get credit, your answers must make sense as English sentences.

Not

(a) A set of vectors is linearly independent if ...

when you put them in a matrix A the equation $Ax=0$ only has the trivial solution

(b) A map $T : V \to W$ is a linear transformation of vectors spaces if ...

$T(v+w) = T(v) + T(w)$

$T(cv) = cT(v)$

(c) A matrix A is invertible if ...

$\det(A) \neq 0$

(d) Let B be an $n \times n$ matrix. Then, a vector $\vec{v}$ is an eigenvector of A if ...

$Av = \lambda v$ for some real number λ

(e) A set of vectors $\mathfrak{B} = \{\vec{v}_1, \ldots, \vec{v}_k\} \subset V$ is a basis for the vector space V if ...

It's a set of LI vectors that spans

(f) A matrix C is an orthogonal matrix if ...

$C^T = C^{-1}$ (or its columns are orthonormal vectors)

(g) A a Markov chain is ...

A set of unit vectors and a matrix whose columns are unit vectors such that

$x_{k+1} = Ax_k$

(h) Let D be a square matrix. Then, the characteristic polynomial of D is ...

$\det(D - \lambda I)$

(i) The rank of a matrix is ...

The dimension of the column space.

(j) The least squares solution to the matrix equation $A\vec{x} = \vec{b}$ is ...

- a solution to $A^TAx = A^Tb$
- The closest solvable problem $Ax = b$

(2) (35 points total, 5 points each) For each question, explain your process and write clearly. All answers must be fully justified, especially answers to yes or no questions.

(a) Let $A_1 = \begin{pmatrix} 1 & 2 & -4 & -4 \\ 2 & 4 & 0 & 0 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 3 & 6 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 5 \\ 2 \\ 5 \\ 5 \end{pmatrix}$. Find all solutions to the matrix equation $A_1\vec{x} = \vec{b}$ or show that no solutions exist.

$$\begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix}$$

System is inconsistent.

(b) Let $A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 0 & -1 & 2 \end{pmatrix}$. Show that the columns of A_2 are either linearly dependent or linearly independent. What does this say about the dimension of *Col A*? Does this imply anything about the dimension of *Nul A*? If so, what and why?

$$A_2 \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$ no pivot in last column $\Rightarrow$ LD

$\dim(Col(A)) = \#$ of pivots $= 2$

$\dim(Nul(A)) = \#$ of free variables $= 1$

because of the rank-nullity theorem

(c) Let $A_3 = \begin{pmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \\ 3 & 6 \end{pmatrix}$. Find a basis for $Nul\ A_3$. What is the rank of A_3? Is A_3 invertible?

$$A_3 \sim \begin{pmatrix} 1 & 5 \\ 0 & -7 \\ 0 & -13 \\ 0 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A_3 has no free variables

$\Rightarrow Nul(A_3) = \{0\}$.

$rank(A_3) = 2$, A_3 is not invertible b/c it's not square.

(d) Let $A_4 = \begin{pmatrix} 3 & -1 & 5 \\ 2 & 1 & 3 \\ 0 & -5 & 1 \end{pmatrix}$. Find a basis for *Row* A_4. What is the rank of A_4? What is the dimension of *Nul A*?

Nbx

$$A_4 \sim \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & 3 \\ 0 & -5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & -1 \\ 0 & -5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 8/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{pmatrix}$$ Basis for $Row(A) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 8/5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1/5 \end{pmatrix} \right\}$

$rank(A) = 2$, $\dim(Nul(A)) = 3 - 2 = 1$

(e) Let $A_5 = \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & 2 \end{pmatrix}$. Find a basis for $Col\ A_5$. What is the rank of A_5?

$$A_5 \sim \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 0 & -2 & 10 & -4 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & \boxed{4} \end{pmatrix}$$

$$\text{Basis} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \right\}, \quad \text{rank} = 3$$

(f) Let $A_6 = \begin{pmatrix} 13 & -5 & 2 \\ -5 & 13 & 2 \\ 2 & 2 & 5 \end{pmatrix}$. Compute the determinant of A_6. Is A invertible?

$$2 \begin{vmatrix} -5 & 2 \\ 13 & 2 \end{vmatrix} - 2 \begin{vmatrix} 13 & 2 \\ -5 & 2 \end{vmatrix} + 5 \begin{vmatrix} 13 & -5 \\ -5 & 13 \end{vmatrix} =$$

$$= 2\left(-10-26\right) - 2\left(26+10\right) + 5\left(169+25\right) =$$

$$= -72 - 72 + 5 \cdot 194 = 970 - 144 = 826$$

A is invertible

(g) Let $A_7 = \begin{pmatrix} \frac{13}{6} & -\frac{5}{6} & \frac{1}{3} \\ -\frac{5}{6} & \frac{13}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{10}{6} \end{pmatrix}$. The eigenvalues of this matrix are $1, 2$ and 3. Find all the eigenvectors of A_7. Is A_7 diagonalizable? If so, give the diagonalization.

$$A_7 - I = \begin{pmatrix} 1/6 & -5/6 & 1/3 \\ -5/6 & 7/6 & 1/3 \\ 1/3 & 1/3 & 4/6 \end{pmatrix} \sim \begin{pmatrix} 7 & -5 & 2 \\ -5 & 7 & 2 \\ 2 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ -5 & 7 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 12 & 12 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow V_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$A_7 - 2I = \begin{pmatrix} 1/6 & -5/6 & 1/3 \\ -5/6 & 1/6 & 1/3 \\ 1/3 & 1/3 & -2/6 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & 2 \\ -5 & 1 & 2 \\ 2 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & 2 \\ 0 & -24 & 12 \\ 0 & 12 & -6 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -5 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} +1 \\ +1 \\ 2 \end{pmatrix}$$

$$A_7 - 3I = \begin{pmatrix} -5/6 & -5/6 & 1/3 \\ -5/6 & -5/6 & 1/3 \\ 1/3 & 1/3 & -8/6 \end{pmatrix} \sim \begin{pmatrix} -5 & -5 & 2 \\ -5 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -4 \\ 0 & 0 & -18 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow V_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(3) (10 points) Let $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Compute the reduced singular value decomposition of B. Does B have a trivial or non-trivial null space? What is the rank of B?

$$B^TB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^TB - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \Rightarrow \det(B^TB - \lambda I) = (1-\lambda)[(1-\lambda)^2 - 1]$$
$$= (1-\lambda)[\lambda^2 - 2\lambda]$$
$$\Rightarrow \lambda = 0, 1, 2 \Rightarrow \sigma = 0, 1, \sqrt{2}$$

$$B^TB - I = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$B^TB - 2I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow z_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$B^TB - 0I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$V = (v_1\ v_2\ v_3) = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\frac{Bv_1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 2/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{Bv_2}{1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

B has a nontrivial null space, it's already in row-reduced form and $Nul(B) = Span\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$

Rank B = # of nonzero singular values = 2

(b) Find the pseudo-inverse of B.

$B^+ = V_r D^{-1} U_r^T$, where V_r, U_r are the parts of V, U associated to non zero singular values.

$$U_r = U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$V_r = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad B^+ = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(4) (10 points) Let Q be an $n \times n$ orthogonal matrix and A an $n \times m$ matrix. Show that
NEX A and QA have the same singular values.

Let $A = U \Sigma V^T$

Then $QA = QU \Sigma V^T = U' \Sigma V^T$ is a SVD for QA,

where $U' = QU$ is the matrix of left singular vectors.

But this shows that the nonzero singular values of QA coincide with those of A

Since QA and A also have the same size, they will also have the same numbero of zero singular values.

(5) (10 points) Let C be a 3×3 symmetric matrix with orthogonal diagonalization given by $C = PDP^{-1}$ where the columns of P are $\{\vec{p}_1, \ldots, \vec{p}_n\}$ and the nonzero entries of the matrix D are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ where $r < n$. Let $\mathfrak{B}$ denote the basis of eigenvectors of C.

(a) What is the change of basis matrix from the standard basis to $\mathfrak{B}$? What is the change of basis matrix from $\mathfrak{B}$ to the standard basis (do not just state this as an inverse of another matrix)?

NET

$\underset{e \leftarrow B}{P} = \left([p_1]_e \;\cdots\; [p_n]_e \right)$, If $\mathcal{E}$ is the standard basis,

$$\underset{\mathcal{E} \leftarrow B}{P} = \left([p_1]_{\mathcal{E}} \;\cdots\; [p_n]_{\mathcal{E}} \right) = \left(p_1 \cdots p_n \right) = P$$

$$\underset{B \leftarrow \mathcal{E}}{P} = \left(\underset{\mathcal{E} \leftarrow B}{P} \right)^{-1} = P^{-1} = P^{T} \text{ b/c } P \text{ is orthogonal.}$$

Note that $\mathfrak{B} = \{p_1, \ldots, p_n\}$ by definition of the orthogonal diagonalization.

NEX (b) What is $[C]_{\mathfrak{B}}$? Justify your answer.

$$[C]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} [C]_{\mathcal{E}} \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = P^{-1} C P = D$$

(6) (5 points) Let

NER

$$D = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

D has a QR decomposition given by

$$D = QR = \begin{pmatrix} \frac{1}{2} & \frac{3\sqrt{5}}{10} & -\frac{\sqrt{6}}{6} \\ -\frac{1}{2} & \frac{3\sqrt{5}}{10} & 0 \\ -\frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{\sqrt{6}}{6} \\ \frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{\sqrt{6}}{3} \end{pmatrix} \begin{pmatrix} 2 & 1 & \frac{1}{2} \\ 0 & \sqrt{5} & \frac{3\sqrt{5}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix}$$

Using the QR factorization, find the least squares solution to $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{pmatrix} 2 \\ -3 \\ -2 \\ 0 \end{pmatrix}$$

Using QR, the least squares solution is

$\hat{x} = R^{-1}Q^T B$.

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1/2 & 1 & 0 & 0 \\ 0 & \sqrt{5} & \frac{3\sqrt{5}}{2} & 0 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & -1/\sqrt{6} \\ 0 & \sqrt{5} & 0 & 0 & 1 & -\frac{3\sqrt{5}}{\sqrt{6}} \\ 0 & 0 & 1 & 0 & 0 & 2/\sqrt{6} \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 & 1 & -1/\sqrt{5} & 2/\sqrt{6} \\ 0 & 1 & 0 & 0 & 1/\sqrt{5} & -3/\sqrt{6} \\ 0 & 0 & 1 & 0 & 0 & 2/\sqrt{6} \end{pmatrix}$$

$$R^{-1} = \begin{pmatrix} 1/2 & -1/2\sqrt{5} & 1/\sqrt{6} \\ 0 & 1/\sqrt{5} & -3/\sqrt{6} \\ 0 & 0 & 2/\sqrt{6} \end{pmatrix}$$

and compute $R^{-1}Q^T b$

(8) (10 points) Consider the following data series:

x	1	2	3	4	5
y	0	2	1	4	5

Suppose we wish to construct a general linear model of the form $y = \beta_1 x + \beta_2 x^3$. What is are design matrix, observation vector and parameter vector for this model? Write down the normal equations for this model but do not solve them.

$$X\beta = y \qquad X = \begin{pmatrix} 1 & 1 \\ 2 & 8 \\ 3 & 27 \\ 4 & 64 \\ 5 & 125 \end{pmatrix}. \qquad y = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \\ 5 \end{pmatrix}$$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

The eqn is $X^T X \beta = X^T y$.

(9) (10 points) Let A be an $m \times n$ matrix. Show that $Nul\ A$ is a subspace of $\mathbb{R}^n$ and that $Row\ A$ is its orthogonal complement.

NEX

To see that $Nul(A)$ is a subspace we need to check $0 \in Nul(A)$ and $cu + dw \in Nul(A)$ if u, w are in it.

But $Nul(A) = \{x \text{ such that } Ax = 0\}$, so

$A0 = 0 \Rightarrow 0 \in Nul(A)$ and if $Au = 0$, $Av = 0$ then

$A(cu + dv) = A(cu) + A(dv) = c(Au) + d(Av) = c \cdot 0 + d \cdot 0 = 0$

$\Rightarrow cu + dv \in Nul(A)$.

$Row(A) = Nul(A)^{\perp}$ because if $A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$ then (rows)

$Ax = \begin{pmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{pmatrix} = 0 \Rightarrow x \in \text{Span}\{a_1, \ldots, a_n\}^{\perp} = Row(A)^{\perp}$