Math 22: Linear Algebra with Applications
Professor Rockmore

Final Exam
Sunday, December 7, 2008

No Calculators. Remember the Honor Code - do all of your own work. Take your time and you'll do fine.

Name:

1.	22 *pts*	
2.	14 *pts*	
3.	8 *pts*	
4.	26 *pts*	
5.	10 *pts*	
6.	14 *pts*	
7.	24 *pts*	
8.	16 *pts*	
9.	20 *pts*	
10.	4 *pts*	
11.	14 *pts*	
12.	3 *pts*	
Total	175 *pts*	

1. **(22 points)** Consider the following system of linear equations

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & = & 1 \\ -x_1 & - & x_2 & + & 4x_3 & - & x_4 & = & 6 \\ -2x_1 & - & 2x_2 & + & 7x_3 & - & x_4 & = & 1 \end{array}$$

(a) **(2 points)** Put them in the form of a matrix/vector equation $A\vec{x} = \vec{b}$.

(b) **(6 points)** Either show the system to be inconsistent or find the solutions.

(c) **(2 points)** If there are solutions, how are the solutions to the associated homogeneous system related to the solutions of the original system.

(d) **(2 points)** What is $dim(Null(A))$?

(e) **(2 points)** Define what is meant by the "rank of a matrix."

(f) **(2 points)** What is $rank(A)$?

(g) **(2 points)** What is $rank(A)$?

(h) **(2 points)** What is $dim(range(A))$?

(i) **(2 points)** What is $dim(row(A))$?

2. **(14 points)**

(a) **(3 points)** Let vectors $\vec{v_1}, \ldots, \vec{v_n}$ be in vector space V. What does it mean for them to be linearly dependent?

(b) **(3 points)** Let $T : V \longrightarrow W$ be a linear transformation between real vector spaces V and W. What does it mean for T to be linear?

(c) **(4 points)** Suppose A is an $n \times n$ real matrix. Give two different conditions for A to be nonsingular (i.e., invertible).

(d) **(4 points)** Let $\vec{b} \in \mathbf{R}^n$ such that the linear system $A\vec{x} = \vec{b}$ is inconsistent.

What do we mean by the statement "$\vec{v} \in \mathbf{R}^m$ is a least squares solution to the system $A\vec{x} = \vec{b}$?" Give an "analytic" answer (i.e., a mathematical statement) as well as a "geometric " definition. (Hint: The latter should involve $Span(Col(A))$.)

3. **(8 points)**
The vectors

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

form a basis for $\mathbf{R}^3$. All vectors are written with respect to the standard basis for $\mathbf{R}^3$.

(a) **(4 points)** Let $\vec{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ be a vector written with respect to the standard basis. Give a matrix/vector expression for the coordinates of $\vec{v}$ with respect to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. **NOTE:** If your expression includes the inverse of some matrix, you need not actually invert the matrix.

(b) **(4 points)** Suppose A is a 3×3 real matrix representing a transformation of $\mathbf{R}^3$ relative to the standard basis. What is the matrix that expresses the transformation relative to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. **NOTE:** If your expression includes the inverse of some matrix, you need not actually invert the matrix.

4. **(26 points)**

(a) **(4 points)** Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 4 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

(b) Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

i. **(2 points)** Compute the determinant of A.

ii. **(2 points)** Does A^{-1} exist? If so, what is it?

iii. **(2 points)** What is the area of image of the unit square in $\mathbf{R}^2$ (it will be some parallelogram) under the transformation A.

iv. **(2 points)** Compute the characteristic equation for A.

v. **(6 points)** The matrix A is diagonalizable - find its eigenvalues and the corresponding eigenvectors.

vi. **(2 points)** Give a geometric interpretation for what A does to $\mathbf{R}^2$. (Hint: use the fact that the eigenvectors form a basis for $\mathbf{R}^2$).

vii. **(2 points)** Suppose B is a 2×2 matrix that effects a rotation in the plane ($\mathbf{R}^2$). Will B have real eigenvectors and real eigenvalues? Why or why not? (Hint: Think geometrically what it means to be an eigenvector!).

viii. **(2 points)** Give an example of a 2×2 rotation matrix.

ix. **(2 points)** What are the eigenvalues of the rotation matrix you gave above?

5. **(10 points)**

(a) **(2 points)** Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

Write down a matrix that **does not** commute with A.

(b) **(2 points)** Suppose that $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a nonzero 2×2 matrix (so it has at least one nonzero entry). Show that there is some vector $\vec{v} \in \mathbf{R}^2$ such that $B\vec{v} \neq \vec{0}$.

(c) **(6 points)** Suppose that B and C are 2×2 matrices with the same eigenvectors $\vec{v}, \vec{w}$ and that $\vec{v}$ and $\vec{w}$ are a basis for $\mathbf{R}^2$. I.e.,

$$B\vec{v} = \lambda_1 \vec{v} \quad B\vec{w} = \lambda_2 \vec{w}; \quad C\vec{v} = \mu_1 \vec{v} \quad C\vec{w} = \mu_2 \vec{w}$$

Show that A and B commute - i.e., $AB = BA$. (Hint: consider what they do to an arbitrary vector $\vec{u} \in \mathbf{R}^2$.

6. **(14 points)**

Let

$$\vec{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{v} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 7 \end{pmatrix}$$

(a) **(2 points)** What is the length of $\vec{u}$?

(b) **(2 points)** What is the cosine of the angle between $\vec{u}$ and $\vec{v}$?

(c) **(2 points)** What is the distance between the points in $\mathbf{R}^4$ represented by $\vec{u}$ and $\vec{v}$.

(d) **(2 points)** Compute the projection of $\vec{v}$ onto $\vec{u}$.

(e) **(4 points)** Write down the matrix that computes that projection of any vector $\vec{w} \in \mathbf{R}^4$ onto $\vec{u}$.

(f) **(2 points)** Use (c) to write $\vec{v}$ as a $\vec{v} = \vec{w_1} + \vec{w_2}$ where $\vec{w_1} \in Span(\vec{u})$ and $\vec{w_2} \in (Span(\vec{u}))^{\perp}$.

7. **(24 points)**

Suppose A is a 3×3 matrix with orthogonal eigenvectors

$$\vec{u}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

and eigenvalues $1, 0, -\frac{2}{3}$ respectively.

(a) **(6 points)** Give the spectral decomposition for A.

(b) **(4 points)** Is A symmetric? Why or why not?

(c) Let $\vec{v} = 6\vec{u}_1 - 2\vec{u}_2 + 3\vec{u}_3$

i. **(4 points)** Compute the projection of $\vec{v}$ onto the subspace spanned by $\vec{u_1}$ and $\vec{u_2}$.

ii. **(2 points)** Let $W = Span\{\vec{u_1}, \vec{u_2}\}$. What is the vector in W that is closest to $\vec{v}$?

iii. **(4 points)** Write down the matrix that takes as input a vector $\vec{w}$ (in standard coordinates) and computes its projection onto the subspace spanned by $\vec{u_1}$ and $\vec{u_2}$.

iv. **(2 points)** Compute Av (in terms of $\vec{u_1}, \vec{u_2}, \vec{u_3}$).

v. **(2 points)** What is $\lim_{n\to\infty} A^n\vec{v}$?

8. **(16 points)**

Let

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{pmatrix}$$

(a) **(8 points)** Use the Gram-Schmidt process to find an orthogonal basis for $Col(A)$.

(b) **(8 points)** Find the QR decomposition of A.

9. **(20 points)**

Suppose that the following table denotes a stock price at times $t_i = 0, 1, 2, 3$.

t_i	0	1	2	3
y_i	1	2	1	5

(a) **(8 points)** Set up and solve the system of normal equations to find the equation of the straight line that best approximates (in a least squares sense) the data.

(b) **(2 points)** Suppose that the data (t, y) represents the price (y) of a stock at time t. What would be the (linear) prediction of the stock price at time $t = 5$?

(c) Suppose we want to find the best quadratic (i.e., second degree) polynomial approximation (in a least squares sense) to the data. I.e., the "best" polynomial $y(t) = \beta_0 + \beta_1 t + \beta_2 t^2$ to approximate the data.

i. **(6 points)** Pose this as a least squares problem – i.e., give the design matrix for solving the associated least squares problem.

ii. **(4 points)** Give the associated collection of normal equations – you can present this as a matrix/vector system of equations. **DO NOT SOLVE THEM.**

10. **(4 points)**

Write down the design matrix for finding the best approximation (in a least squares sense) by a plane $z = \beta_0 + \beta_1 x + \beta_2 y$ for the data

x_i	1	2	3	4
y_i	1	1	-1	0
z_i	3	-2	8	4

I.e., The value z_i is what was observed for a given input of (x_i, y_i).

11. **(14 points)**

(a) **(2 points)** Suppose A represents the matrix of a regular Markov chain. What is the equilibrium distribution and how do you find it?

(b) Let A be the matrix of a discrete dynamical with eigenvalues $\frac{1}{4}$ and $\frac{1}{2}$ with eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

i. **(2 points)** Classify the origin as an attractor, repellor or saddle point.

ii. **(2 points)** In what direction does the trajectory change the fastest?

iii. **(4 points)** If $\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, what is $\vec{x}_1 = A\vec{x}_0$?

iv. **(4 points)** For a positive integer k, what is $\vec{x}_k = A^k\vec{x}_0$?

12. **(3 points)**

Give an example of an application of linear algebra that you found surprising and/or interesting.