MATH22 - Linear Algebra with Applications

Exam II ANSWERS

August 1, 2007

1. (20 points) Tell whether each statement below is true or false. You need not show your work.

(a) Any basis for $\mathbb{P}_2$ (the space of polynomials of degree ≤ 2) has a polynomial of each possible degree, i.e. a polynomial of degree 0, degree 1, and degree 2.

(b) Every subspace of a finite dimensional vector space has a basis.

(c) If A is a 3×5 matrix and rank $A = 2$, then dim Nul $A = 3$.

(d) The null space and the column space of a square matrix A can have a non-trivial intersection.

Answer:

(a) False. One basis of $\mathbb{P}_2$ is $\{1 + t + t^2, t + t^2, t^2\}$, which has no polynomials of degree less than 2.

(b) False. The subspace $\{\mathbf{0}\}$ has no basis.

(c) True. If the rank of A is 2, then the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has 3 free variables, and so dim Nul $A = 3$.

(d) True. If $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, then $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for both the null space and the column space of A.

2. (20 points) Find bases for Nul A and Col A, where

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \end{bmatrix}$$

Answer:

Since the third column of A is twice the second plus the first and since the first and second are not multiples of one another, a basis for Col A is

$$\left\{ \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

This also gives that the general solution to the homogeneous equation has the form

$$\begin{bmatrix} x \\ 2x \\ -x \end{bmatrix}$$

for any arbitrary $x \in \mathbb{R}$, and hence a basis for Nul A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

3. (15 points) Let A be an $n \times n$ matrix. Which of the following are equivalent to the statement: A is an invertible matrix?

(a) A is a product of elementary matrices.

(b) Row $A = \mathbb{R}^n$.

(c) Nul $A = \mathbb{R}^n$.

(d) There is a nonzero vector in $\mathbb{R}^n$ perpendicular to Row A.

(e) There is an $n \times n$ matrix C such that $AC = 0$.

(f) The equation $A\mathbf{x} = \mathbf{0}$ has finitely many solutions for $\mathbf{x} \in \mathbb{R}^n$.

Answer:

(a) This is an equivalent statement. Recall this is how we found the algorithm to find the inverse.

(b) This is an equivalent statement. If A is invertible, $n = \text{rank } A = \dim \text{Row } A$.

(c) This is not an equivalent statement. If every vector in $\mathbb{R}^n$ is in the null space, then the rank of A is $0 \neq n$.

(d) This is not an equivalent statement. If there is a nonzero vector perpendicular to Row A, then Row A has dimension less than n, i.e. A does not have full rank.

(e) This is not an equivalent statement. In fact, it is true for non-invertible matrices as well (put $C = 0$).

(f) This is an equivalent statement. If the homogeneous equation has finitely many solutions, we have by Theorem 2 on page 24 that it has a unique solution, which is true if and only if the matrix A is invertible.

4. (15 points) Suppose A is a 4×4 matrix in echelon form. What is $\det A$ when the pivots of A are:

(a) $\{-1, -2, -3\}$

(b) $\{1, 1, 4\}$

(c) $\{5, -1, 2, 3\}$

Answer:

(a) Since there are three pivots and A is 4×4, $\det A = 0$.

(b) Since there are three pivots and A is 4×4, $\det A = 0$.

(c) Since A is in echelon form, $\det A$ is the product of the pivots, so $\det A = -30$.

5. (15 points) Let D be the derivative operator on $\mathbb{P}_4$. What is the standard matrix for D?

Answer:

For $0 \leq n \leq 4$, the operator D takes t^n to nt^{n-1} and the standard basis for $\mathbb{P}_4$ is $\{1, t, t^2, t^3, t^4\}$, so the standard matrix for D is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. (15 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T(\mathbf{x}) = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix} \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^2$. If U is the unit square, what is the area of $T(U)$? (Recall that the unit square is the square determined by the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.)

Answer:

The area of $T(U)$ is given by

$$\begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix} \{\text{area of } U\} = (8 - (-2)) \{\text{area of } U\} = 10 \cdot 1 = 10.$$

7. (20 points) Let W be a subspace of the finite dimensional vector space V such that $\dim W < \dim V$. Prove that there exists a subspace W' of V such that $V = W + W'$ and $W \cap W' = \{\mathbf{0}\}$. (Recall the sum $W + W'$ is defined in problem 33 on page 225.)

Answer:

Proof. Pick a basis for W, say $A = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, where $\dim W = n$. Since this is a linearly independent set of vectors in V, we may extend this set to a basis of V, say $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}, \dots, \mathbf{v}_m\}$, where $\dim V = m > n$. Then, let $W' = \text{Span } C$, where $C = \{\mathbf{v}_{n+1}, \dots, \mathbf{v}_m\}$. This is clearly a subspace, and by the work in problem 33 on page 225, $W + W' = V$. If $\mathbf{v} \in W \cap W'$, then $\mathbf{v}$ can be written in two ways, namely

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v_n} = \mathbf{v} = a_{n+1}\mathbf{v}_{n+1} + \cdots + a_m\mathbf{v}_m. \tag{1}$$

Subtracting the right hand side from the left hand side, we see that this means

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v_n} - a_{n+1}\mathbf{v}_{n+1} - \cdots - a_m\mathbf{v}_m = \mathbf{0}. \tag{2}$$

Since the basis B is linearly independent, we see that (2) has only the trivial solution $a_1 = a_2 = \cdots = a_m = 0$. Plugging this solution into (1), we see that $\mathbf{v} = \mathbf{0}$. Thus, $W \cap W' = \{\mathbf{0}\}$, and the statement is proven. □

8. (25 points) Let

$$\mathscr{A} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$
$$\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$
$$\mathscr{C} = \left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(a) Write the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathscr{B}}$ in $\mathscr{A}$-coordinates, then in $\mathscr{C}$-coordinates.

(b) If $[\mathbf{x}]_{\mathscr{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, find a matrix expression for $\mathbf{x}$ in $\mathscr{A}$-coordinates.

Answer:

(a) We have

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

and

$$\underset{A \leftarrow B}{P} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$$

Applying these change of coordinate matrices, we have

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathscr{B}} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}_{\mathscr{A}} = \begin{bmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}_{\mathscr{C}}.$$

(b) Here,

$$\underset{A \leftarrow C}{P} = \underset{A \leftarrow B}{P}\,\underset{B \leftarrow C}{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 1 \\ -\frac{5}{3} & 0 \end{bmatrix},$$

and so if $[\mathbf{x}]_{\mathscr{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$[\mathbf{x}]_{\mathscr{A}} = \begin{bmatrix} \frac{4}{3} & 1 \\ -\frac{5}{3} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_1 + x_2 \\ -\frac{5}{3}x_1 \end{bmatrix}.$$

9. (25 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and that $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $T^{-1}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$ and $T^{-1}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

Answer:

It is given that $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and we have

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

From this, we can conclude that

$$T(\mathbf{x}) = \begin{bmatrix} -2 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^2$. Thus,

$$T^{-1}(\mathbf{x}) = -\frac{1}{4} \begin{bmatrix} 2 & -1 \\ 0 & -2 \end{bmatrix} \mathbf{x}$$

and plugging in the two values for $\mathbf{x}$ we see

$$T^{-1}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{2} \end{bmatrix} \qquad \text{and} \qquad T^{-1}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \end{bmatrix}.$$

10. (30 points) Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. For each of the following subsets, either prove that the subset is a subspace of V or give a reason why it is not.

(a) The functions f such that $f(1) = f(2)$.

(b) The functions f such that $f(-x) = -f(x)$.

(c) The functions f such that $f(1) = f(2) + 1$.

Answer:

(a) The zero function is an element of this set as $f(1) = f(2) = 0$ for that function. If two functions f and g are in the set, then

$$(f+g)(1) = f(1) + g(1) = f(2) + g(2) = (f+g)(2),$$

so $f + g$ is in the set. Finally, if $f(1) = f(2)$, then

$$(cf)(1) = cf(1) = cf(2) = (cf)(2)$$

for any real number c. Thus, this set is a subspace of V.

(b) This set is also known as the set of odd functions. The zero function is an odd function as $f(-x) = f(x) = 0$ gives that $-f(x) = 0$. If f and g are odd functions, then

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f(x) + g(x)) = (-(f+g))(x).$$

For any real number c and any odd function f,

$$(cf)(-x) = cf(-x) = c(-f(x)) = -cf(x) = (-cf)(x).$$

Thus, this set is also a subspace of V.

(c) This set is not a subspace as the zero function is not in the set.

11. (0 points) Bonus: It can be shown that, as vector spaces, $L(\mathbb{R}^n, \mathbb{R}^m)$ and $M_{m\times n}$ are isomorphic, i.e. that there exists a bijection φ between them which satisfies

$$\varphi(cS + dT) = c\varphi(S) + d\varphi(T) \tag{3}$$

for every $S, T \in L(\mathbb{R}^n, \mathbb{R}^m)$ and every $c, d \in \mathbb{R}$. Note, as a bijection, φ is both one-to-one and onto and hence invertible. One way to find φ is to find its value on a basis and apply (3) together with the properties of a basis to see that this determines the value of φ everywhere else. In particular, φ sends a basis of $L(\mathbb{R}^n, \mathbb{R}^m)$ to a basis of $M_{m\times n}$. Use this information (and one theorem from your book) to find a basis for $L(\mathbb{R}^n, \mathbb{R}^m)$.

Answer:

By Theorem 10, page 83, every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T(\mathbf{x}) = A_T\mathbf{x}$ for a unique $m \times n$ matrix A_T and any vector $\mathbf{x} \in \mathbb{R}^n$. This gives a natural bijection between the two vector spaces. Further, it is easy to show that

$$\varphi(cS + dT) = A_{cS+dT} = cA_S + dA_T$$

for any $S, T \in L(\mathbb{R}^n, \mathbb{R}^m)$ with associated matrices A_S and A_T and any $c, d \in \mathbb{R}$. Let $M_{i,j}$ be the $m \times n$ matrix with a 1 in the i, j-th position and 0's elsewhere. This is clearly a basis for $M_{m\times n}$, so taking the transformations which correspond to these matrices under φ, i.e. applying φ^{-1} to this set, gives a basis for $L(\mathbb{R}^n, \mathbb{R}^m)$. In particular, if $T_{i,j}$ is the linear transformation for which $T_{i,j}(\mathbf{x}) = M_{i,j}\mathbf{x}$, then the set $\{T_{i,j} : 1 \le i \le m \text{ and } 1 \le j \le n\}$ is a basis for $L(\mathbb{R}^n, \mathbb{R}^m)$. $T_{i,j}$ can also be described as the unique linear transformation for which

$$T_{i,j}(\mathbf{e}_k) = \begin{cases} \mathbf{e}_j & \text{if } k = i \\ \mathbf{0} & \text{otherwise.} \end{cases}$$