Math 22: Linear Algebra. PRACTISE MIDTERM 2 — ANSWERS

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1. a) +8, b) $\lambda = 1$ with $\mathbf{v} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$, and $\lambda = 6$ with $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

2. Col A has basis $\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$. Row A has basis $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \\ 6 \end{bmatrix}$.

   Proof: basis for H must be lin indep vectors, which also lie in V. However, no more than dim V vectors can be lin indep in V. QED.

3.
$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

   LU is useful since once performed, $A\mathbf{x} = \mathbf{b}$ can be solved for vectors $\mathbf{b}$ with only $O(N^2)$ effort, where N is typical size of matrix.

4. a) Nul A, so it's a subspace, write out proof of Thm 2 (p. 227).

   b) basis is the one vector $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. dim $W = 1$.

5. (a) $\begin{bmatrix} -2 \\ -7 \\ 8 \end{bmatrix}$

(b) There are 2 basis vectors, so you know $[\mathbf{x}]_{\mathcal{B}}$ must have 2 components, call them c_1 and c_2.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$$

must be satisfied, since this what the $\mathcal{B}$-coords of $\mathbf{x}$ mean. This is just a linear equation which we solve by row reduction of augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(R.E.F.)}$$

It is consistent (otherwise $\mathbf{x}$ would not be in H). The unique solution is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

6. (a) False: $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because its elements (2-component vectors) do not even come from $\mathbb{R}^3$ (the set of 3-component vectors). It is not even a subset.

(b) a linear transformation that is both one-to-one and onto

(c) False. $A\mathbf{x}$ is a unique object, so it cannot both be $\lambda_1\mathbf{x}$ and $\lambda_2\mathbf{x}$, which it would have to be if an eigenvector for both eigenvalues.

(d) Write the 3 polynomials in the standard basis, to get

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\} \quad \text{stack as cols, reduce to get} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

All 3 pivots, so we have 3 linearly-independent vectors in $\mathbb{R}^3$, so they form a basis. (You could also instead have said they span $\mathbb{R}^3$). $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$ so the original polynomials also form a basis for $\mathbb{P}_2$.