Math 22: Linear Algebra. MIDTERM 2

2 hrs, no calculators. Please answer all six questions. Answer on this sheet. Your NAME:

1. [11 points]

   (a) Compute (without using row swaps) the LU decomposition of $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -4 & -3 & 2 & 0 \\ 6 & 2 & 0 & 1 \end{bmatrix}$

(b) Counting from the left as usual, which is the *first* column of A that can be written as linear combination of the previous ones, and why?

(c) Let B be any lower triangular matrix with non-zero entries on the diagonal. Prove that the inverse of B exists and is also lower triangular. [Hint: elementary row operations].

2. [12 points]

(a) Find the real eigenvalues (if they exist) and multiplicities of $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

(b) Find the real eigenvalues (if they exist) and multiplicities of $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

(c) Find the eigenvalues (which are all real) and multiplicities of $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(d) Find a basis for the eigenspace associated with the above double eigenvalue. What is its dimension?

3. [10 points]

(a) True/false: Two eigenvectors with the same eigenvalue are always linearly independent?

(b) What is the rank of a 5×3 matrix if a basis for its null space contains only one vector?

(c) True/false: Given a $n \times n$ matrix A, if $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ then A must have at least one real eigenvalue?

(d) True/false: The set $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0 \right\}$ is a subspace of $\mathbb{R}^2$?

(e) Explain why a $n \times n$ matrix can have at most n eigenvalues.

4. [7 points] Compute the determinants of the following matrices: [Hint: in each case one method is much easier than the other]

(a) $$\begin{bmatrix} 2 & 0 & 6 \\ 0 & 7 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 9 \\ 3 & 7 & 29 \end{bmatrix}$$

5. [11 points] The matrix A has been converted to reduced echelon form as follows

$$A = \begin{bmatrix} -2 & -4 & 1 & 0 & 4 \\ 0 & 0 & 0 & 3 & 3 \\ 1 & 2 & 2 & 1 & 5 \\ 3 & 6 & 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Write down a basis for the column space of A:

(b) Write down a basis for the null space of A:

(c) What is the dimension of the subspace consisting of all possible vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{x}$?

(d) What is the dimension of the subspace consisting of all solutions to the equation $A\mathbf{x} = \mathbf{0}$?

(e) Explain why the first 3 rows of the R.E.F. of A form a basis for Row A.

6. [9 points]

(a) Does the set $\{1+t^2, t+t^2, t-t^2\}$ form a basis for the vector space of all polynomials of the form $a+bt+ct^2$? Explain what criteria you tested, and if each test failed or passed.

(b) The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ form a basis $\mathcal{B}$ for $\mathbb{R}^3$. If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$.

(c) Prove that for any basis for $\mathbb{R}^n$ the coordinate mapping $\mathbf{x} \to [\mathbf{x}]_{\mathcal{B}}$ is one-to-one.