Math 22, Exam I

April 22, 2010

NAME:

This is a closed book exam and you may not use a calculator. Use the space provided to answer the questions and if you need more space, please use the back of the exam making sure to write a note in the space provided that you have more work elsewhere that you would like me to grade. You must SHOW ALL WORK and be neat. If you have any questions, do not hesitate to ask.

Good luck!

Remember the honor code – do all of your own work.

1. Let

$$A = \begin{pmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

a. If consistent, solve the system $A\mathbf{x} = \mathbf{b}$ and write its solutions in parametric form. If it is not consistent, say so.

$$\begin{pmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{pmatrix} \xmapsto{R1 \leftrightarrow R3} \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{pmatrix} \xmapsto{R_3 \mapsto R_3 - 5R_1} \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{pmatrix}$$

$$\xmapsto{R_3 \mapsto R_3 + 7R_2} \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{pmatrix}$$

NO! INCONSISTENT BECAUSE $-29 \neq 0$.

b. Solve the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Same row operations as above lead to

$$\begin{pmatrix} 5 & 8 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 3 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} x_1 + 3x_3 &= 0 \\ x_1 - x_3 &= 0 \end{aligned}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}.$$

c. Is the system $A\mathbf{x} = \mathbf{c}$ consistent for all $\mathbf{c} \in \mathbb{R}^3$? Explain.

NO! FOR $\bar{c} = \bar{b}$ is INCONSISTENT BY THE ARGUMENT FROM a), ABOVE.

2. Consider the three vectors

$$\begin{pmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{pmatrix}$$

a. Find h such that the matrix is the augmented matrix of a consistent linear system.

$$\begin{pmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 5R_1} \begin{pmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{pmatrix}$$

The condition is $h + 15 \neq 0$, so $h \neq -15$.

b. Find h such that the three columns of the above matrix are linearly independent.

– There is no such h. We have $3 > 2$ vectors in $\mathbb{R}^2$ so there is always a linear dependence.

– or: By the arguments at a). if

$h+15 \neq 0 \Rightarrow x_3$ is free variable.

if $h = -15 \Rightarrow x_2$ is free variable

At any rate, there is always a free variable so the homogeneous system has a nontrivial solution.

3. Compute the determinants of the following matrices:

a.

$$\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} = 5 \times 2 - 3 \times 4 = -2$$

b.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

$$\text{ROW } 1 \quad = - \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = -(3-8) + (2-6)$$

$$= 5 - 4 = 1.$$

c.

$$\begin{pmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & 0 & 2 \end{pmatrix}$$

$$\text{COLUMN } 2 \quad = -3 \begin{vmatrix} 4 & -7 & 3 & -5 \\ 0 & 2 & 0 & 0 \\ 5 & 5 & 2 & -3 \\ 0 & 9 & 0 & 2 \end{vmatrix} = \quad \text{ROW } 2$$

$$= -3 \cdot 2 \cdot \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & 0 & 2 \end{vmatrix} \quad : \text{ROW } 3$$

$$= -3 \cdot 2 \cdot 2 \begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix} = -12(8-15) =$$

$$= (-12)(-7) = 84.$$

4. Find the inverses of the following matrices.

a.

$$\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}.$$

$$-\frac{1}{2}\begin{pmatrix} 2 & -4 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 3/2 & -5/2 \end{pmatrix}$$

b.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \longmapsto \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \longmapsto \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -5 & 0 & -2 & 1 \end{array}\right)$$

$$\longmapsto \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -2 & 1 \end{array}\right) \longmapsto \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right) \longmapsto \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 16 & -7 & 4 \\ 0 & 1 & 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right).$$

So, $A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 5 & -2 & 1 \\ -4 & 2 & -1 \end{pmatrix}$

c.

$$\begin{pmatrix} 1 & 2 & -1 \\ 5 & -2 & 4 \\ -1 & 2 & 3 \end{pmatrix}$$

FOR c). SIMILAR ARGUMENTS GIVE

$$A^{-1} = \frac{1}{60}\begin{pmatrix} 14 & 8 & -6 \\ 19 & -2 & 9 \\ -8 & 4 & 12 \end{pmatrix}.$$

5. Let

$$A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$$

a. Find the LU decomposition of A.

$$A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 5R_2}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix}. \qquad \longmapsto \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} =: U$$

b. Let

$$B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Write B as a product of elementary matrices.

$$\underset{E_3}{\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}} \quad \underset{E_2}{\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}} \quad \underset{E_1}{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}} \qquad \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_3 E_2 E_1 B = I_2 \implies B = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

c Write B^{-1} as a product of elementary matrices.

$$B^{-1} = E_3 E_2 E_1 = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

6. Answer the following questions by true or false:

a. The inverse of an elementary matrix is an elementary matrix.

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b. The following matrix is invertible

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

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(R3 = R2 + R1 SO DETERMINANT IS 0)

c. Any linear system of equations whose coefficient matrix is of type 3×4 has a free variable.

T.

d. I like linear algebra.

EXTRA CREDIT

7. Let $T : \mathbb{R}^3 \longmapsto \mathbb{R}^2$ be the linear map given by

$$T(x_1, x_2, x_3) = (3x_2 - x_3, 2x_1 + x_2 + 3x_3).$$

a. What is the domain of T?

$\mathbb{R}^3$

b. What is the co-domain of T?

$\mathbb{R}^2$

c. What is the standard matrix for T?

$$\begin{pmatrix} 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

d. Is T onto? Why or why not?

YES. The matrix of T is

$\sim \begin{pmatrix} 2 & 1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$ so it has 2 pivots.

e. Is T one-to-one? Why or why not?

No! The matrix has only 2 pivots so "there is ~~a~~ pivot in every column." is false.

8. Let $T : \mathbb{R}^2 \longmapsto \mathbb{R}^3$ be the linear transformation given by

$$T(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad T(\mathbf{e}_2) = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

a. Compute

$$T\begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

$$T\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3T(e_1) + 5T(e_2) = 3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 29 \end{pmatrix}$$

b. Is $T : \mathbb{R}^3 \longmapsto \mathbb{R}^2$ given by $T(x_1, x_2, x_3) = (x_1 + 2x_3, x_1 + |x_2|)$ linear? Explain why or why not.

$T(1,1,1) = (3,2)$, $T(-1,-1,-1) = (-3,0)$

So $T(-1,-1,-1) \neq -T(1,1,1)$. The map is not linear.

c. Suppose that A and B are $n \times n$ matrices such that both A and AB are invertible. Is B invertible?

① YES! $\det(AB) \neq 0$ because AB is invertible

$\det(AB) = \det A \cdot \det B \neq 0 \Rightarrow \det B \neq 0.$

So B is INVERTIBLE.

② ELSE: $C := (AB)^{-1} A$, observe that

$CB = ((AB^{-1}) A) B = (AB)^{-1} AB = I_2$

So B is INVERTIBLE. ITS INVERSE is C.

③ ELSE: $B = A^{-1} \cdot AB$ = PRODUCT OF TWO INVERTIBLE MATRICES

SO IT is INVERTIBLE.