Lecture 7: Linear transformations

Danny W. Crytser

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Example

We write

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

because $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a vector in $\mathbb{R}^2$.

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Example

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Example

The rule $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x)$ is a function from $\mathbb{R}$ to $\mathbb{R}$. The range is $[-1, 1] \subset \mathbb{R}$, because $\sin(x)$ only takes on values between -1 and 1, and it takes on all those values.

Properties of the product $A\mathbf{x}$

Definition

Let A be an $m \times n$ matrix and $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then $A\mathbf{x}$ is a vector in $\mathbb{R}^m$ equal to the linear combination of the columns of A using the entries of $\mathbf{x}$ as entries.

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These two facts show that the function $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $\mathbf{x} \mapsto A\mathbf{x}$ is *linear* in a certain sense.

Linear transformations: definition

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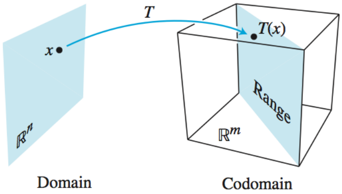

FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

Linear transformations: example

Example

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The map $\mathbf{x} \to 2\mathbf{x}$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ for any space n. (Just a consequence of the algebraic properties of vector addition and scalar multiplication.)

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Example

The map $\mathbf{x} \to 2\mathbf{x}$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ for any space n. (Just a consequence of the algebraic properties of vector addition and scalar multiplication.) In fact, this is the same function you get when you multiply vectors in $\mathbb{R}^n$ by the matrix

$$A = \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 2 \end{bmatrix}.$$

More examples with linear transformations

Example

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 4 \end{bmatrix}$ and define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$

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Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. Then $T(\mathbf{u}) = \begin{bmatrix} 1+2 \\ 1-3-8 \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \end{bmatrix}$.

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Probably not, although you can find a good approximation using graphical and numerical means.

Solving equations with linear transformations is awesome

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$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 3 & 2 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & -7 & -5 & -11 \end{bmatrix}$$

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Thus $x = 2/7 + z/7$, $y = 11/7 - 5z/7$, and z is free. A solution is $(3/7, 6/7, 1)$. Thus $T(3/7, 6/7, 1) = \left[\begin{array}{c} 5 \\ 4 \end{array}\right]$.

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You can also answer questions about whether or not there is more than one solution to $T(\mathbf{x}) = \mathbf{b}$ using the same method, by writing things in terms of an augmented matrix and then using an echelon form.

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We have our system in reduced echelon form:

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Thus $x = 2/7 + z/7$, $y = 11/7 - 5z/7$, and z is free. A solution is $(3/7, 6/7, 1)$. Thus $T(3/7, 6/7, 1) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

You can also answer questions about whether or not there is more than one solution to $T(\mathbf{x}) = \mathbf{b}$ using the same method, by writing things in terms of an augmented matrix and then using an echelon form.

Remark

Solving equations involving linear equations easier than solving other eqns.

Solving equations with linear transformations: another example

Define a map $T(\mathbf{x}) = A\mathbf{x}$ where

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Thus $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \left(\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}\right)$. T takes a vector in $\mathbb{R}^3$, thought of as a point in three-dimensional space, and drops it on to the xy-plane.

Example: rotations

There is a special collection of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ which rotate the plane.

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A rotation matrix generally looks like $\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$, where t is the angle you are rotating by counterclockwise.

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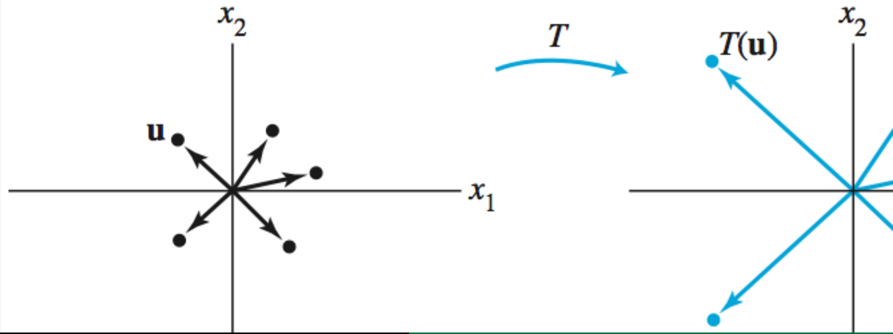

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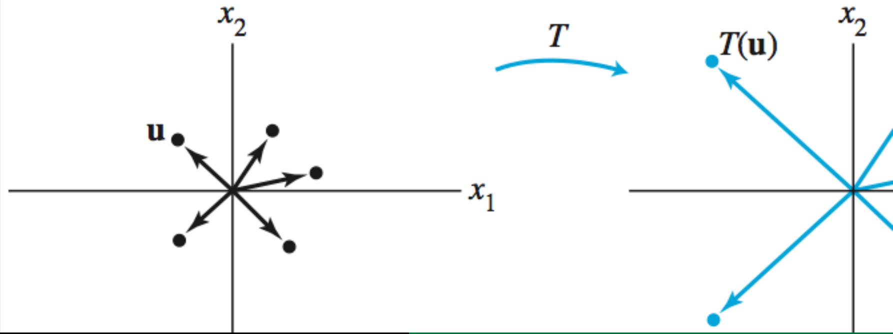

Properties of linear transformations

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Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation between vector spaces.

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Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation between vector spaces. Then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for any vectors $\mathbf{u}, \mathbf{v}$ and scalars c, d.

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Let's prove that $T(\mathbf{0}) = \mathbf{0}$ and $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for linear transformation.

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Linear transformations and linear combinations

We have seen that for a linear transformation we always have $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all vectors and scalars.

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If $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$ are vectors and $c_1, \ldots, c_p$ are scalars, and T is a linear transformation with domain $\mathbb{R}^n$, then

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You can prove this just by repeating the proof of the case for $p = 2$. This fact will be very useful to us: it will enable us to write any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ in the form $T(\mathbf{x}) = A\mathbf{x}$ for a unique matrix A.