Lecture 6: Linear independence

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April 2, 2014

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   We will define a property, called *linear independence*, which is useful for studying this question.
2. We will describe geometrically what it means for a set containing one or two vectors to be linearly independent.
3. We will give some alternate ways of studying linearly independent and dependent sets, and some basic theorems.

Linear independence: definition

Definition

A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors in $\mathbb{R}^n$ is called **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = x_2 = \ldots = x_p = 0$.

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Linear independence: examples

Example

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Linear independence: example with row reduction

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Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, $\mathbf{v}_3 = (2, 1, 0)$.

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Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, $\mathbf{v}_3 = (2, 1, 0)$. Let's determine using the row reduction algorithm if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

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where we have omitted the constant column because we know it's all zeros.

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$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$$

is a linear dependence relation for the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Linear independence and $A\mathbf{x} = \mathbf{0}$, I

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Let A be an $m \times n$ matrix, columns $\mathbf{a}_i, i = 1, \ldots, p$. Any linear dependence relation among the columns

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are *not* linearly independent. No matter how you row-reduce there is no way that every column of A can contain a pivot, so there will always be free variables in the solution to $A\mathbf{x} = \mathbf{0}$. Since the solution to $A\mathbf{x} = \mathbf{0}$ is not unique, the columns are linearly dependent.

Dependence among one vector

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Proof.

The weights $x_1 = 1$, $x_2 = x_3 = \ldots = x_p = 0$ are a non-trivial solution to

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In fact, this works for sets containing the zero vector along with any number of other vectors. We dignify this *fact* with the lofty title of theorem:

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Example

The set $\{\mathbf{0}, (1, 0, 2), (0, 0, 1), (7, 2, 0)\}$ is linearly dependent.

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Remark

The stuff after "in fact" just says that you can look at the vectors "in order" and test to see if each is a linear combination of the vectors that preceded it, and then j can be the first index where you can actually write the linear combination.

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So we wrote, as is always possible, one of the vectors in the linearly dependent set as a linear combination of the others.

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Also we can't say anything in the case when $n = p$: could be linearly independent (e.g. $\{(1,0),(0,1)\} \subset \mathbb{R}^2$) or linearly dependent (e.g. $\{(1,1),(2,2)\} \subset \mathbb{R}^2$).

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