Lecture 5: Homogeneous, inhomogeneous, solution sets. Applications

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Today's lecture

1. Finish up with homogeneous equations, learning when they have a nontrivial solution.
2. Describe the solution set homogeneous equation as the span of a finite set of vectors.
3. Describe the solution set of an inhomogeneous equation.
4. Use systems of linear equations to model economic behavior.
5. Use systems of linear equations to model street traffic.

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$$A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$. Is there a nontrivial solution to $A\mathbf{x} = \mathbf{0}$?Reduce the augmented matrix to echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

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Is every column a pivot column? No, so the solution is not unique, and there is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$. In this example, $\mathbf{x} = (-2, 1, 0)$ is a nontrivial solution.

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Thus the solution set is

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Now write $x + 2y = 3$, solve for the basic variable x, to get $x = 3 - 2y$, y free. Solutions look like
$(x, y) = (3 - 2y, y) = (3, 0) + y(-2, 1)$, where y is any number.

Solutions of inhomogeneous equations

Theorem

Let $A\mathbf{x} = \mathbf{b}$ *be an inhomogeneous matrix equation, where* A *is a* $m \times n$ *matrix and* $\mathbf{b} \neq \mathbf{0}$*. Suppose that* $\mathbf{p}$ *is a particular solution to the system. Consider the homogeneous system* $A\mathbf{z} = \mathbf{0}$*. Then every other solution* $\mathbf{w}$ *of* $A\mathbf{x} = \mathbf{b}$ *has the form*

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Suppose that $A\mathbf{w} = \mathbf{b} = A\mathbf{p}$. Then we can subtract to obtain

$$A(\mathbf{w} - \mathbf{p}) = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

So $\mathbf{w} - \mathbf{p} = \mathbf{v}$ for some $\mathbf{v}$ a solution of $A\mathbf{z} = \mathbf{0}$ □

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The solutions set is then $\{\mathbf{p} + \mathbf{v} : A\mathbf{v} = \mathbf{0}\}$.

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The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 6 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Example: parametric form for inhomogeneous soutions

Write all solutions to

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. Thus the general parametric form of the solution to the inhomogeneous is

$$\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

where t is allowed to be any real number.

Change

Now we're going to look at some applications of linear systems: economics and street traffic.

Application: Economics

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The first example of this we shall see comes from economics. Many economists divide the economy of a city, province, or nation into sectors. Examples of these could include: coal, electricity, steeletc. You measure the output of a sector in dollars. The sectors use their own output and the output of the other sectors: steel needs coal, producing coal requires electricity, electric plants need steel, etc.

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Economics

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.0	.4	.6	C
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We denote the output of the three sectors by p_C, p_E, p_S.

Economics, ctd.

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$$p_C = .4p_E + .6p_S$$

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Rewriting these in linear form we obtain

Economics, ctd.

$$
\begin{aligned}
p_C - .4p_E - .6p_S &= 0 \\
-.6p_C + .9p_E - .2p_S &= 0 \\
-.4p_C - .5p_E + .8p_S &= 0
\end{aligned}
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Economics, ctd.

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$$\mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

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Streets can be used as an example of networks:

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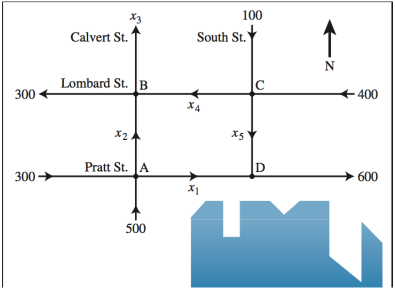

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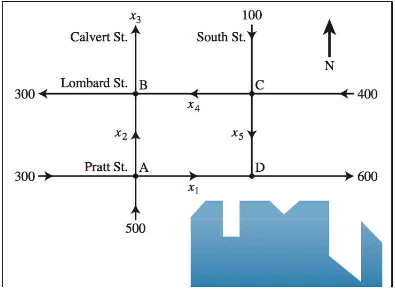

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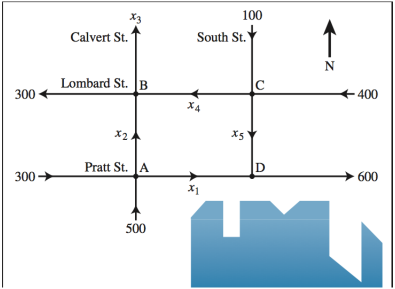

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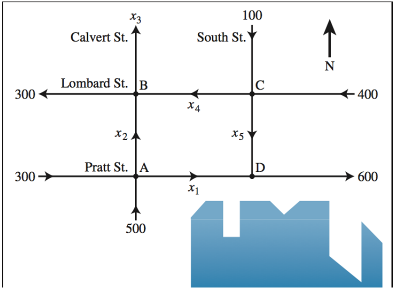

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Node (intersection)	Flow in		Flow out
A	$300 + 500$	$=$	$x_1 + x_2$
B	$x_2 + x_4$	$=$	$300 + x_3$
C	$100 + 400$	$=$	$x_4 + x_5$
D	$x_1 + x_5$	$=$	600

Traffic flow, ctd.

We have the balanced flow equations

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We also need that the flow into the system $(500 + 300 + 100 + 400)$ equals the flow out $(300 + x_3 + 600)$.

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We also need that the flow into the system $(500 + 300 + 100 + 400)$ equals the flow out $(300 + x_3 + 600)$. We simplify and combine all of this into a system of equations:

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad &= 800 \\ x_2 - x_3 + x_4 \qquad &= 300 \\ x_4 + x_5 &= 500\,. \\ x_1 \qquad\qquad\qquad + x_5 &= 600 \\ x_3 \qquad\qquad &= 400 \end{aligned}$$

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If we solve this system with row reduction we get the solution set

$$\begin{cases} x_1 &= 600 - x_5 \\ x_2 &= 200 + x_5 \\ x_3 &= 400 \\ x_4 &= 500 - x_5 \\ x_5 & \text{is free} \end{cases}$$

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Again, real world constraints make the solution set smaller. x_4 cannot be negative because there cannot be a negative number of cars passing through a branch. So $0 \leq x_5 \leq 500$.

What this is used for

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