Lecture 4: $A\mathbf{x} = \mathbf{b}$ and solution sets

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Today's lecture

We saw in the previous lecture that solving systems of linear equations is equivalent to solving certain vector equations

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Today we are going to further compress our notation, writing the sum on the left side of the equation $(*)$ as a matrix-vector product $A\mathbf{x}$. We will see that solving such **matrix equations** is equivalent to solving systems of linear equations, and that we can extract much useful information about the solution set by studying the matrix A.

Review of matrices

Remember that an m-by-n matrix A is a rectangular array of numbers

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The columns of A are

$$\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

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$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n.$$

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in the form $A\mathbf{x}$ for some choice of matrix A and some choice of weights $\mathbf{x} = (x_1, x_2, x_3)$?

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Note that the position of the vector as a column is the same as the position of the weight in the weight vector.

Matrix equations

Writing linear combinations of vectors in the form $A\mathbf{x} = \mathbf{b}$ gives us another way to write systems of linear equations.

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which can also be written as the **matrix equation**

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If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ vectors in $\mathbb{R}^m$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

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which has the same solutions as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

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is consistent. This is row-equivalent to

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

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Properties of the matrix-vector product, ctd.

Proof.

We are going to prove that if $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$ is a $m \times n$ matrix, $\mathbf{u} = (u_1, u_2, \dots, u_n)$ is in $\mathbb{R}^n$ and c is a scalar, then

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$$cu_1\mathbf{a}_1 + cu_2\mathbf{a}_2 + \dots + cu_n\mathbf{a}_n = c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \dots + u_n\mathbf{a}_n).$$

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On the left hand side (LHS), we have $A(c\mathbf{u}) = A(cu_1, cu_2, \dots, cu_n)$ by definition of the scalar multiple $c\mathbf{u}$But then for the LHS we get $A(c\mathbf{u}) = cu_1\mathbf{a}_1 + cu_2\mathbf{a}_2 + \dots + cu_n\mathbf{a}_n$. We can write this as

$$cu_1\mathbf{a}_1 + cu_2\mathbf{a}_2 + \dots + cu_n\mathbf{a}_n = c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \dots + u_n\mathbf{a}_n).$$

But within the parentheses we have $A\mathbf{u}$, so that

$$A(c\mathbf{u}) = c(A\mathbf{u}).$$

□

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Definition

A system of linear equations is *homogeneous* if the constants are all 0. Otherwise *inhomogeneous*.

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$$A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0s \\ 0 \end{bmatrix}$$

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Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$. Is there a nontrivial solution to $A\mathbf{x} = \mathbf{0}$?Reduce the augmented matrix to echelon form:

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Describing planes

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Thus the solution set is

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Parametric vector equations

In the previous example, every solution to the system

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$$(x, y, z) = c(2, 1, 0) + d(4, 0, 1)$$

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Now write $x + 2y = 3$, solve for the basic variable x, to get $x = 3 - 2y$, y free. Solutions look like $(x, y) = (3 - 2y, y) = (3, 0) + y(-2, 1)$, where y is any number.

Solutions of inhomogeneous equations

Theorem

Let $A\mathbf{x} = \mathbf{b}$ *be an inhomogeneous matrix equation, where* A *is a* $m \times n$ *matrix and* $\mathbf{b} \neq \mathbf{0}$. *Suppose that* $\mathbf{p}$ *is a particular solution to the system. Consider the homogeneous system* $A\mathbf{z} = \mathbf{0}$. *Then every other solution* $\mathbf{w}$ *of* $A\mathbf{x} = \mathbf{b}$ *has the form*

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Suppose that $A\mathbf{w} = \mathbf{b} = A\mathbf{p}$. Then we can subtract to obtain

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So $\mathbf{w} - \mathbf{p} = \mathbf{v}$ for some $\mathbf{v}$ a solution of $A\mathbf{z} = \mathbf{0}$ □

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The solutions set is then $\{\mathbf{p} + \mathbf{v} : A\mathbf{v} = \mathbf{0}\}$.

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$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \dots + a_nx_n.$$

Row-column products (dot products)

We are going to simplify computations of matrix-vector products $A\mathbf{x}$.

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We can repeat this operation with each row of a matrix.