Lecture 3: Vector equations

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We have seen that the solution sets to linear equations can often be described as lines in the plane. In today's lecture we will make this precise and extend it to cover systems with more free variables. This will allow us to visually describe the solution set of a system of linear equations. The notion of a vector will allow us to simplify our description of solution sets, and the algebraic relationships between vectors will reflect properties of the solution set.

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(The computer puts the w-column in the fourth column instead of the first. Henceforth, the w-column is the fourth column.) We will transform this to echelon form, use the echelon form to check if it is consistent, and then, if it is consistent, further transform this to reduced echelon form.

Ex. of RRA, ctd.

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Step 1: Identify the first pivot column.

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Step 3 Now we clean out the two entries beneath the first pivot position. Add -2 times the first row to the second row, add -1 times the first row to the third row.

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Now we clean out the column above the pivot in the third row, third column, subtracting the third row from each other row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 5 \\ 0 & -1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 0 & 1 & 5 \\ 0 & -1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

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Now the matrix is in reduced echelon form. We write down the system of linear equations corresponding to this matrix:

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$$\begin{aligned} x + w &= 2 \\ y &= -3 \\ z &= 1 \end{aligned}$$

We rewrite x in terms of the free variable w appearing in the first equation, thus describing the solution set.

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If you're into 4-dimensional space, this is a line in 4-dimensional space.

Vectors

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The following are not vectors:

$$\{1, 2\} = \{2, 1\}, \text{ the concept of melancholy }, \infty$$

Notation for vectors

The notation for vectors varies somewhat depending on what context you're working in. Sometimes a vector is represented with parentheses:

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$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}.$$

Rows are not the same as vectors. A row is a matrix with one row. A vector is a matrix with one column. For our purposes the word vector will always mean a column vector, even if we sometimes write them horizontally, in which case we will use parentheses to show that we mean to denote a vector.

Operations with vectors: addition

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Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ be two vectors.

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$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}.$$

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That is, cv is the vector obtained by multiplying each entry of $\mathbf{v}$ by c.

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If $\mathbf{v} = (v_1, \dots, v_n)$ is a vector, what is the scalar product of $\mathbf{v}$ by 0. $0\mathbf{v} = (0, 0, \dots, 0)$, for any vector $\mathbf{v}$.

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Definition

The space of all ordered lists of two real numbers is denoted by $\mathbb{R}^2$ (read "r-two"). The space of all ordered lists of three real numbers is denoted by $\mathbb{R}^3$ (read "r-three"). The space of all ordered lists of n real numbers is denoted by $\mathbb{R}^n$ (read "r-n").

Visualizing vectors in $\mathbb{R}^2$

Adding vectors in $\mathbb{R}^2$ has a nice visual interpretation.

Remark

We can visualize vectors in $\mathbb{R}^2$ as points in the xy-plane, where $\mathbf{v}$ is the point in the plane with x-coordinate v_1 and y-coordinate v_2.

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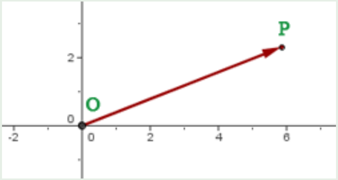
Example

Here is the vector $\mathbf{v} = (6, 2.1)$ represented in the plane as the point P



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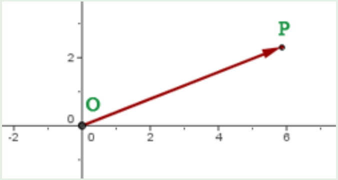

Geometric picture of $\mathbb{R}^2$

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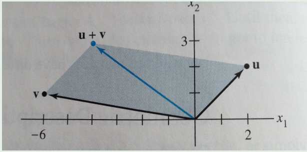

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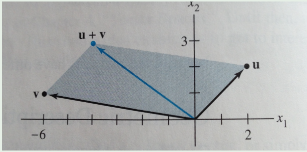

Photo credit: my camera phone.

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Any solution to this system is a set of weights $(x_1, \dots, x_p)$ with $\sum_{i=1}^n x_i\mathbf{a}_i = \mathbf{b}$.

Span$\{\mathbf{v}\}$

The set of all vectors that we can obtain as linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$ has a special name: the **span** of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$

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$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

for *some* choice of weights $c_1, \ldots, c_p \in \mathbb{R}$.

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