Row reduction and echelon forms

Danny W. Crytser

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1. The leading entry of the first row is the 7 in the second column.
2. The leading entry of the second row is the 1 in the first column.
3. The third row does not have a leading entry–they are all zero.

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4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Example

Check out this matrix

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$$\begin{bmatrix} 1 & -3 & 0 & 7 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Same question.

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The matrices

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

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are row-equivalent: we can add -2 times the first row of A to the second row of A to obtain B.

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$$\begin{aligned} x + y &= 2 \\ 2x + 2y &= 4 \end{aligned}$$

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Let A be a matrix. Then there is a unique matrix U in reduced echelon form which is row-equivalent to A.

If A is a matrix, then we call the unique U in this theorem the *reduced echoelon form of A*.

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If A is a matrix, then we call the unique U in this theorem the *reduced echoelon form of A*. Our goal in this section is to develop a technique for systematically transforming matrices, via elementary row operations, to reduced echelon form.

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$U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The only leading one in the reduced echelon form is in the first column and first row. So the only pivot position of A is in the first row and first column, and the only pivot column of A is the first column.

Example: The row reduction algorithm

Let

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

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Step 1: Begin with the leftmost nonzero column. This is a pivot column. There is a pivot position at the top of this column.

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$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

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Step 3: Use row replacement to create zeros in positions below the pivot positions. (Add multiples of the first row to the rows containing zeros in the first column.)

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Now below the first pivot position there are no nonzero entries.

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Step 4: Now we ignore the row containing the first pivot position and every row above it, and we repeat steps $1-3$ to the matrix that remains.

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$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Now we ignore the two top rows and look at the matrix that remains. The leftmost nonzero column is the fourth column. We repeat steps $1-3$: all that we have to do is interchange the third and fourth rows.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Now the matrix is in *echelon form*. This completes steps $1-4$.

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We transform this into reduced echelon form: all the leading entries should be equal to 1, and the columns containing the leading entries only have 0s, except for the leading entries.

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Step 5: Beginning with the rightmost pivot, create zeros above each pivot position. If necessary, first scale rows containing pivot positions to make the pivots equal 1.

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We scale the third row by $\frac{-1}{5}$, then add 6 times this new third row to the second row and 9 times the new third row to the first row.

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The next pivot position is in the second row, second column.

Row reduction algorithm, ctd.

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The next pivot position is in the second row, second column. We scale the second row by $\frac{1}{2}$ to get a 1 in the pivot position, then add -4 times this new second row to the first row to eliminate the 4 above the pivot position.

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This matrixis in *reduced* echelon form: the leading entries are 1, no nonzero entries above the leading entries.

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Forward and backward phases of the row reduction algorithm

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2. Step 5 is called the *backward phase* of the algorithm. It converts the echelon form into a reduced echelon form.

Solutions to linear systems

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Example

The augmented matrix

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

corresponds to the system

$$\begin{aligned} -3x_2 - 6x_3 + 4x_4 &= 9 \\ -x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ -2x_1 - 3x_2 + 3x_4 &= -1 \\ x_1 + 4x_2 + 5x_3 - 9x_4 &= -7 \end{aligned}$$

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We converted

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solutions to linear systems, ctd.

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This echelon form corresponds to the system

$$x_1 - 3x_3 = 5$$

$$x_2 + 2x_3 = -3$$

$$x_4 = 0$$

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When dealing with the system

$$\begin{aligned} x_1 - 3x_3 &= 5 \\ x_2 + 2x_3 &= -3 \\ x_4 &= 0 \end{aligned}$$

we sort the variables x_1, x_2, x_3, x_4 into two categories.

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2. The other variables are called *free variables*: in this case: x_3 is free.

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Basic variables: x_1, x_2, x_4. Free variables: x_3. We write the basic variables in terms of the free variables, and that describes the solution set:

$$\begin{cases} x_1 = & = 5 + 3x_3 \\ x_2 = & -3 - 2x_3 \\ x_3 & \text{is free} \\ x_4 = & 0 \end{cases}$$

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The solution set of both systems is described by:

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Existence/uniqueness with echelon forms

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Example: existence and uniqueness with echelon forms

Example

Suppose that we have a linear system whose augmented matrix we have reduced to the echelon form

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 7 & -1 & 1 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

Does there exist a solution to the system?

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In this case, is the solution unique?