Lecture 13: Vector spaces

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April 18, 2014

Today's lecture

1. We will discuss subspaces of $\mathbb{R}^n$: subsets in which you can add and scale vectors.
2. We will talk about bases
3. Column space and null space of a matrix; finding bases for these spaces.
4. Vector spaces
5. Subspaces of vector spaces

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2. If $\mathbf{u}, \mathbf{v} \in H$, then $\mathbf{u} + \mathbf{v} \in H$. (*Closed under addition.*)
3. For each $\mathbf{u} \in H$ and each scalar $c \in \mathbb{R}$, the vector $c\mathbf{u} \in H$ (*Closed under scalar multiplication.*)

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4. In $\mathbb{R}^3$, any plane passing through the origin is a subspace.

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so we obtain a new linear combination with weights kd_i. This shows that the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of $\mathbb{R}^n$. Sometimes we'll call $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ the *subspace* spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

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The vector is a linear combination of the columns of A, so it belongs to the column space of A.

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System is consistent, so $\mathbf{b} \in \text{Col}\, A$.

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adjective

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□

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We are going to be interested in throwing out as many vectors as we can without changing the span.

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Remark

It is not enough for a set to be linearly independent in order for it to be a basis, nor is it enough for a set to be spanning. It has to be *both* linearly independent and spanning.

Examples of bases

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The most important example of a basis is one we've worked with a lot.

Definition

The columns of any invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$: they are linearly independent and spanning by the Invertible Matrix Theorem. In particular the columns of I_n, the $n \times n$ identity matrix, form a basis for $\mathbb{R}^n$ called the **standard basis** for $\mathbb{R}^n$.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Example: not a basis

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2. $S_2 = \{(1, 1, 0)\}$ is a linearly independent set in H, because it only has one element and that element is nonzero. However, S_2 is *not* a basis for H. Why?

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$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 3 & 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

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The solution is $x_1 = 0, x_2 = -x_4, x_3 = x_4, x_4$ free.

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The solution is $x_1 = 0, x_2 = -x_4, x_3 = x_4, x_4$ free. Thus

$$\text{Nul}\, A = \text{span}\{(0, -1, 1, 1)\}$$

and $\{(0, -1, 1, 1)\}$ is a basis for the null space of A.

Finding bases for $\operatorname{Col} A$

Computing a basis for $\operatorname{Col} A$ is straightforward:

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So far we've defined subspaces of $\mathbb{R}^n$ as things where you can add and scale vectors. There are in fact many examples of sets with naturally defined addition and scalar multiplication.

Vector spaces: definition

Definition

A **vector space** is a nonempty set V of objects, called *vectors*, which we can add and multiply by scalars and all the following axioms hold whenever $\mathbf{u}, \mathbf{v} \in V$, $c, d \in \mathbb{R}$:

1. $\mathbf{u} + \mathbf{v} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. there is a zero vector $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. there is $-\mathbf{u} \in V$ with $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. the scalar multiple $c\mathbf{u} \in V$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

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We know that when $V = \mathbb{R}^n$ all these properties of addition and scalar multiplication hold. The idea is that any set with addition and scalar multiplication which plays "this nice" will enjoy all the nice properties of $\mathbb{R}^n$.

Example of vector spaces: differentiable functions

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We can define **pointwise addition and scalar multiplication** on this set by

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x)c(f(x))$$

for $f, g \in V$ and $c \in \mathbb{R}$. It is a fact from calculus that if f and g are differentiable then $f+g$ is differntiable and cf is differentiable. Thus $f+g \in V$ and $cf \in V$.

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Examples of vector space: matrices

Example

Let m and n be integers and define

$$M = M_{m,n} = \{A = [a_{ij}] : A \text{ is a } m \times n \text{ matrix}\}.$$

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Noteice that you get a different vector space for every choice of (m, n): you can only add vectors of the same size. Thus there is the vector space of 2×2 matrices, the vector space of 3×2 matrices, etc.

Examples of vector spaces: polynomials

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Let $n \geq 1$ be an integer and define

$$\mathbb{P}_n = \{a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n : a_0, \ldots, a_n \in \mathbb{R}\}.$$

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Thus $\ltimes$ is the set of all polynomials with degree $\leq n$. You can add polynomials

$$(a_0 + a_1 t + + \ldots + a_n t^n) + (b_0 + b_1 t + \ldots + b_n t^n) = (a_0 + b_0) + (a_1 + b_1) t + \ldots +$$

You can multiply polynomials by scalars

$$c(a_0 + a_1 t + + \ldots + a_n t^n) = (ca_0) + (ca_1) t + \ldots + (ca_n) t^n.$$

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Checking that all the vector space axioms hold is kinda boring but within your powers (hah, ugh).

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of the vectors in S. (You can modify this notation to allow for infinite sets: just require that all but finitely many of the weights c_i are 0.)

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As H satisfies *all* the three properties, it is a subspace.

A non-example of a subspace: differentiable functions

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So H violates every one of the three conditions a subset must satisfy in order to be a subspace. Other examples, such as the ones you will encounter on homework, might only violate one or two.

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Non-example: matrices

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Let M_2 denote the vector space of 2×2 matrices. Then

$$H = \{A \in M_2 : \det A \neq 0\},$$

the set of all invertible 2×2 matrices, is *not* a subspace of M_2.

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Example: polynomials

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Let $\mathbb{P}_3$ be the vector space of polynomials with degree ≤ 3 (add and scale coefficients).

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Thus H is the span of a set of vectors in $\mathbb{P}_3$, which means that H is automatically a subspace.

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