Math 22 – Linear Algebra and its applications

- Lecture 9 -

Instructor: Bjoern Muetzel

GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, Th, **Sun 2-4 pm** in KH 229

- **Tutorial:** Tu, Th, **Sun 7-9 pm** in KH 105

- **Midterm 1**: **Monday Oct 7** from **4-6 pm** in **Carpenter 013**

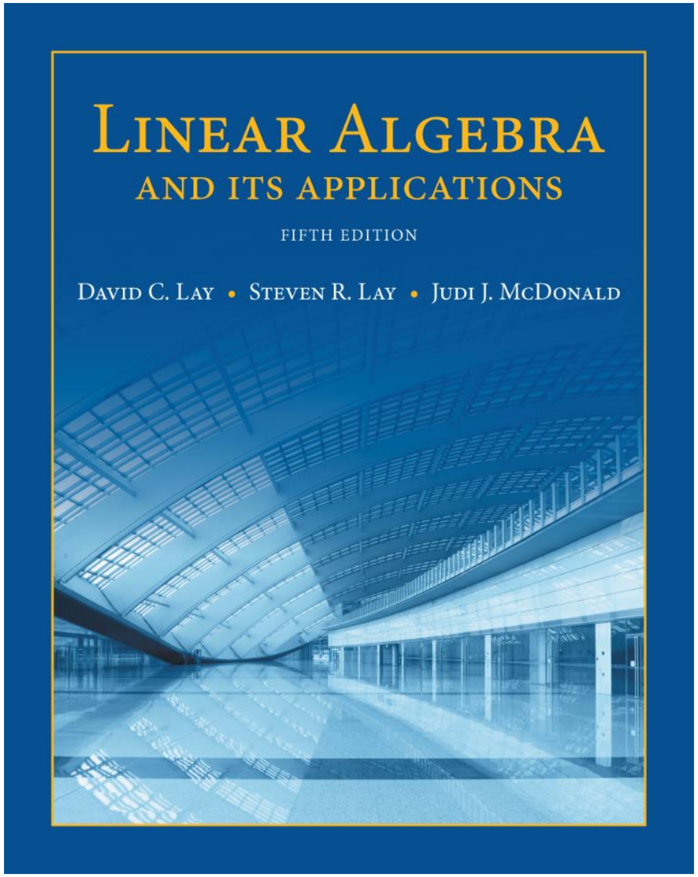
  Topics: till this **Thursday** (included)

  You can find the **practice exam** online

2 Matrix Algebra

2.1

MATRIX OPERATIONS



- **<u>Summary</u>:** The **composition** of **linear maps** is equal to the **multiplication** of the **associated standard matrices**.

MATRICES

- If A is an $\boldsymbol{m} \times \boldsymbol{n}$ matrix, then A *has* $\boldsymbol{m}$ **rows** and $\boldsymbol{n}$ **columns**.

  The entry in the ***i*-th row** and ***j*-th column** of A is denoted by $\boldsymbol{a_{ij}}$ and is called the **(*i*, *j*)-entry** of A.

- Each column of A is a vector in $\mathbb{R}^m$. The **j-th column vector** is denoted by $\boldsymbol{a_j}$

Column j

Row i

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A = [a_1 \quad a_2 \quad \ldots \quad a_n]$$

$\uparrow$ $\mathbf{a}_1$ $\quad$ $\uparrow$ $\mathbf{a}_j$ $\quad$ $\uparrow$ $\mathbf{a}_n$

MATRICES

- **Note:** The **diagonal entries** of an $m \times n$ matrix $A = \left[a_{ij} \right]$ are a_{11}, a_{22}, a_{33}, …, and they form the **main diagonal** of A.
  A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are all zero.

- **Example:** I_n, the $n \times n$ identity matrix and **0** the $n \times n$ **zero** matrix.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

SUM AND SCALAR MULTIPLICATION

- **Definition:** If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries in A and B.

- **Note:** The sum $A + B$ is **defined only** when A and B are the **same size**.

- **Definition:** If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose entries are r times the corresponding entries in A.

SUM AND SCALAR MULTIPLICATION

- **Theorem 1:** Let A, B, and C be matrices of the same size, and let r and s be scalars.

  a. $A+B=B+A$

  b. $(A+B)+C=A+(B+C)$

  c. $A+0=A$

  d. $r(A+B)=rA+rB$

  e. $(r+s)A=rA+sA$

  f. $r(sA)=(rs)A$

  This is easily verified as all operations are performed **entrywise**.

SUM AND SCALAR MULTIPLICATION

Example: For $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$

1.) Find $A+B$ and $A+C$.

2.) Find $3C$ and $A+2B$.

- **Solution:**

COMPOSITION OF LINEAR MAPS

- **Theorem:** Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations with standard matrices B and A, respectively. Then the **composition** $S \circ T: \mathbb{R}^n \to \mathbb{R}^p$ is again a **linear transformation.**

- **Proof:**

- **What is the standard matrix** C **of the composite transformation** $S \circ T: \mathbb{R}^n \to \mathbb{R}^p$ **?**

We know by **Theorem 10** that for the basis $\{e_1, e_2, \ldots, e_n\}$ in $\mathbb{R}^n$

$$C = [S \circ T(e_1) \ldots S \circ T(e_n)] .$$

COMPOSITION OF LINEAR MAPS

- **Can we express C in terms of A and B?**

- To this end we translate our problem into matrix notation:

- The image $T(\mathrm{x})$ of a vector $\mathbf{x}$ is $B\mathbf{x}$.
- To get the image $S \circ T(\mathrm{x}) = \mathrm{S}(\mathrm{T}(\mathrm{x}))$ of $\mathbf{x}$, we must apply S to $B\mathbf{x}$.
- This is equal to multiplying A with the vector $B\mathbf{x}$. Hence

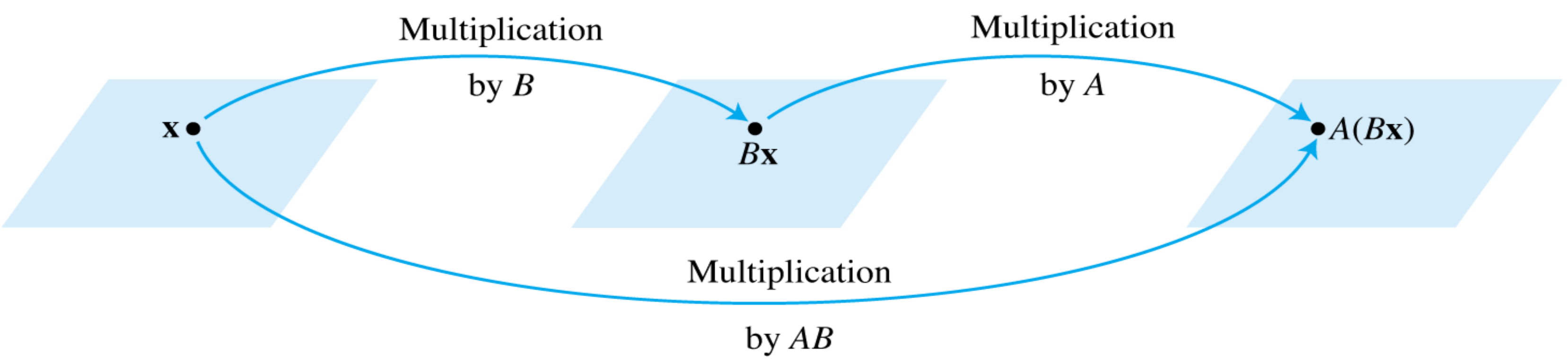
$$S \circ T(\mathrm{x}) = A(B\mathbf{x}).$$

- Thus $A(B\mathbf{x})$ is obtained from x by the ***composition of two linear mappings*.**

MATRIX MULTIPLICATION

- **<u>Goal:</u> Define** matrix multiplication AB, such that AB is the matrix C of the composite transformation $S \circ T$.

$$\boxed{A(B\mathrm{x}) = (AB)\mathrm{x} = C\mathrm{x}}$$



Multiplication by AB.

MATRIX MULTIPLICATION

- We can find C in terms of A and B by looking at the images of the basis $\{e_1, e_2, \ldots, e_n\}$ in $\mathbb{R}^n$. We know that

$$C = [S \circ T(e_1) \ldots S \circ T(e_n)] \ .$$

We have:

MATRIX MULTIPLICATION

Therefore we **define**:

- **Definition:** If A is a $p \times m$ matrix, and if B is an $m \times n$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_n$, then the product AB is the matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_n$. That is the matrix product is

$$AB = [Ab_1, Ab_2 \ldots, Ab_n].$$

- **This way multiplication of matrices corresponds to composition of linear transformations.**

MATRIX MULTIPLICATION

- **Example:** Compute AB, where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

- **Solution:**

MATRIX MULTIPLICATION

Row-column rule for computing $C=AB$

- If A is an $m \times k$ matrix, and if B is a $k \times n$ matrix then the product $C=AB$ is defined.

- In this case the

entry c_{ij} in **row** $\boldsymbol{i}$ and **column** $\boldsymbol{j}$ of $C=AB$,

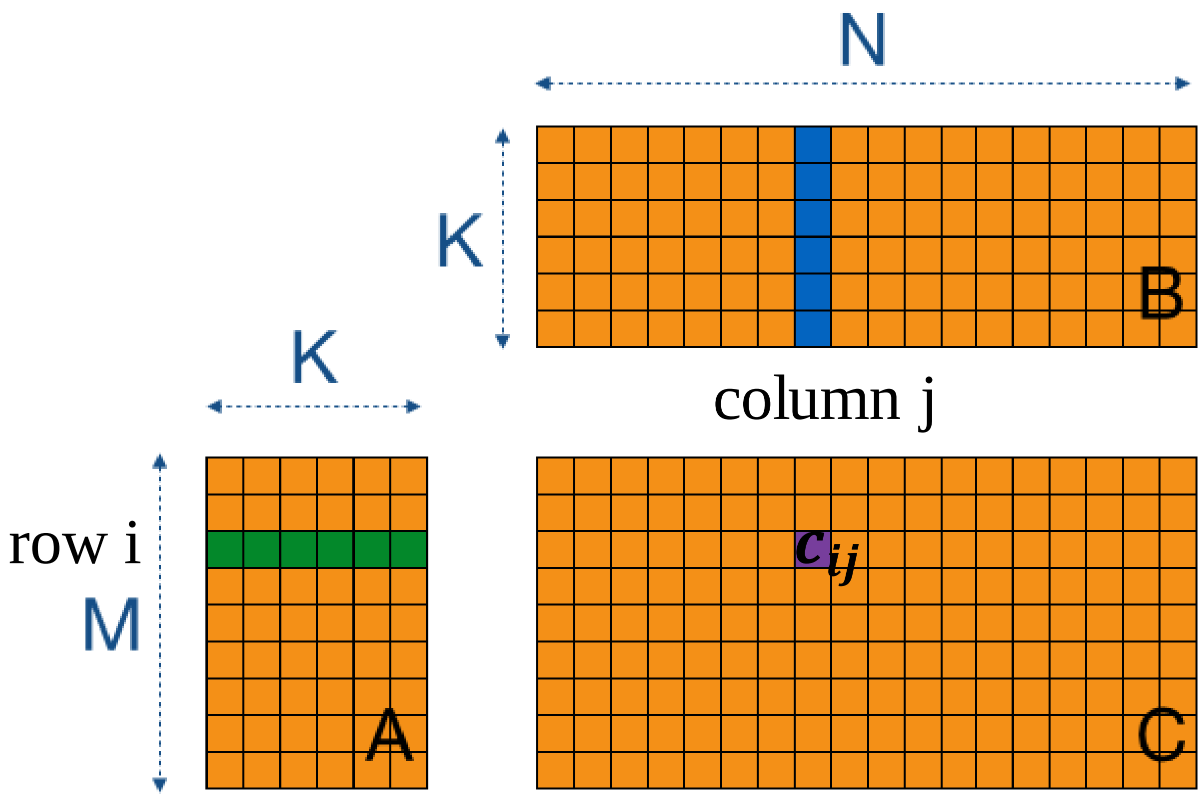
is the **sum** of the **products** of corresponding

entries from **row** $\boldsymbol{i}$ of A and **column** $\boldsymbol{j}$ of B.

If $c_{ij} = (AB)_{ij}$, then

$$\boxed{(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}}.$$

Row-column rule for computing C=*AB*



PROPERTIES OF MATRIX MULTIPLICATION

- **Theorem 2:** Let A, B and C be matrices that have sizes for which the indicated sums and products are defined. Then

  a. $A(BC) = (AB)C$ (associative law of multiplication)

  b. $A(B+C) = AB + AC$ (left distributive law)

  c. $(B+C)A = BA + CA$ (right distributive law)

  d. $r(AB) = (rA)B = A(rB)$ for any scalar r

  e. $I_m A = A = AI_n$ (identity for matrix multiplication)

PROPERTIES OF MATRIX MULTIPLICATION

- **Proof:** (a) follows from the fact that matrix multiplication corresponds to composition of linear mappings and as the composition of mappings is associative.
- (b)-(e) are easily verified using the definition.

PROPERTIES OF MATRIX MULTIPLICATION

- **Definition:** If $AB = BA$, we say that A and B **commute** with one another.

- **Warnings:**
  1. In general, $AB \neq BA$.
  2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is ***not*** true in general that $B = C$.
  3. If a product AB is the zero matrix, you ***cannot*** conclude in general that either $A = 0$ or $B = 0$.

POWERS OF A MATRIX

- **Definition:** If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k}$$

- We set $A^0 = I_n$, the identity matrix.

THE TRANSPOSE OF A MATRIX

- **Definition:** Given an $m \times n$ matrix *A*, the **transpose** of *A* is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of *A*.

Theorem 3: Let *A* and *B* denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar *r*, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

Example: For $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$, find A^2 and B^T.

- **Solution:**

PROPERTIES OF MATRIX MULTIPLICATION

- **Exercise:** For each of the following statements find different nontrivial 2×2 matrices, that satisfy them.

  1.) $AB = BA$.

  2.) $AB \neq BA$.

  3.) $AB = AC$, but $B \neq C$.

  4.) $AB = 0$ but $A \neq 0$ and $B \neq 0$.