Math 22 –
Linear Algebra and its applications

- Lecture 8 -

Instructor: Bjoern Muetzel

GENERAL INFORMATION

- **<u>Office hours:</u>** Tu 1-3 pm, **Th**, Sun 2-4 pm in KH 229
- **<u>Tutorial:</u>** Tu, **Th**, Sun 7-9 pm in KH 105

- **<u>Attention:</u>** This **Thursday** the **x-hour** will be a **lecture**:

  Section 1: 12:15 - 1:05 pm in Kemeny 007

  Section 2: 1:20 - 2:10 pm in Kemeny 007

  office hour will start at 2:15 pm.

- **<u>Midterm 1</u>**: **Monday** Oct 7 from **4-6 pm** in **Carpenter 013**

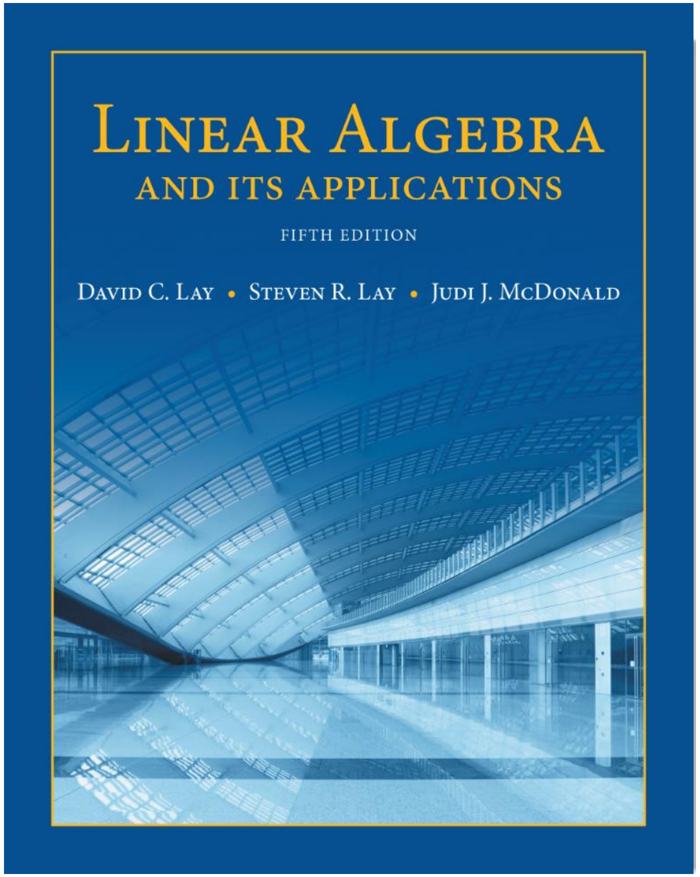
  Topics: till this **Thursday** (included)

  You can find the **practice exam** online

1 Linear Equations in Linear Algebra

1.9

THE MATRIX OF A LINEAR TRANSFORMATION



- **Summary**:

1.) In **finite dimensions linear** and **matrix transformations** are the **same**.

2.) We find **conditions** that show us when a **linear transformation** maps **onto** the whole codomain and when it is **one-to-one**.

GEOMETRIC LINEAR TRANSFORMATIONS OF $\mathbb{R}^2$

A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is completely determined by the two images $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$:

GEOMETRIC LINEAR TRANSFORMATIONS OF $\mathbb{R}^2$

GEOMETRIC LINEAR TRANSFORMATIONS OF $\mathbb{R}^2$

- **Example:** The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

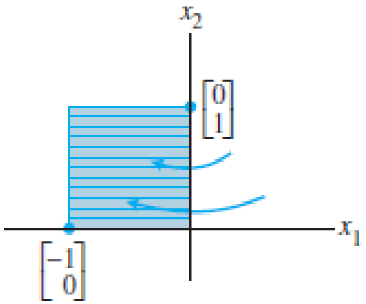
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is a rotation around the origin with angle t.

GEOMETRIC LINEAR TRANSFORMATIONS OF $\mathbb{R}^2$

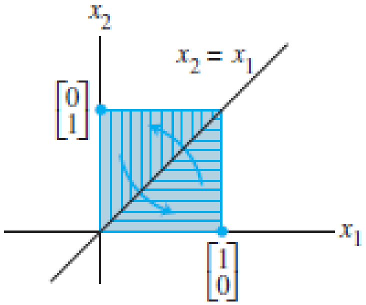
Example: Reflections:

Reflection through the x_2-axis



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

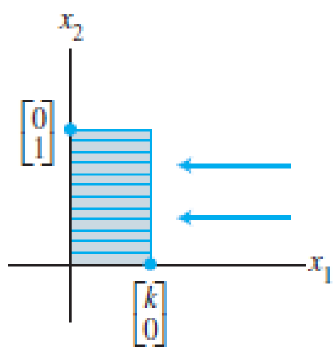
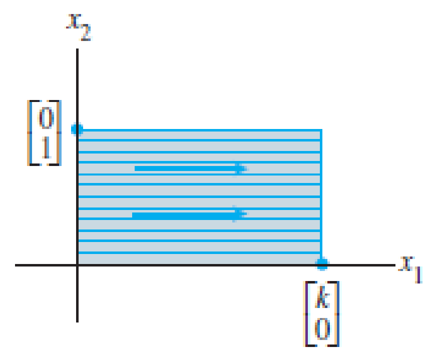
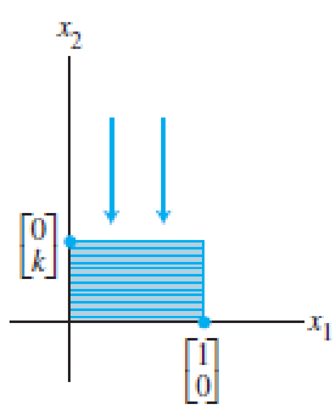
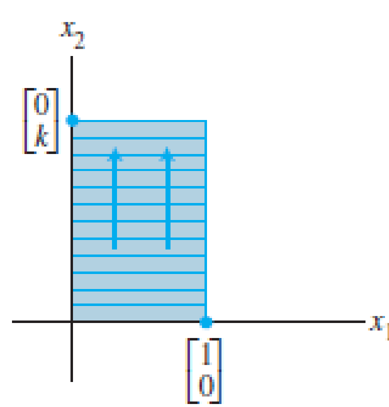
Reflection through the line $x_2 = x_1$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

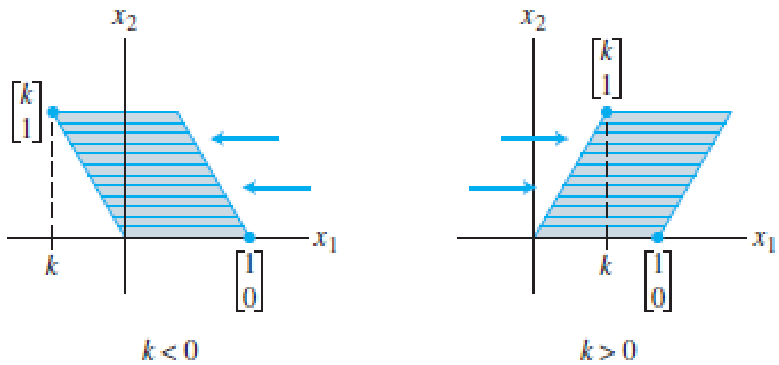
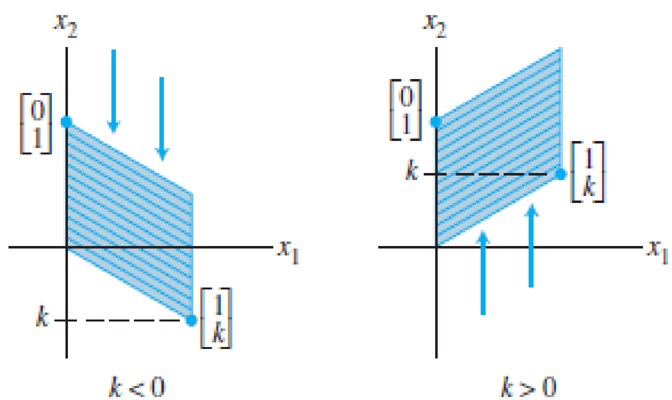
GEOMETRIC LINEAR TRANSFORMATIONS OF $\mathbb{R}^2$

TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$  $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$  $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

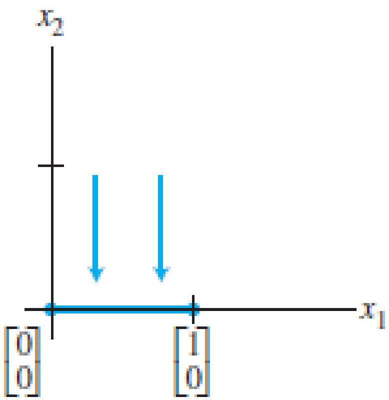
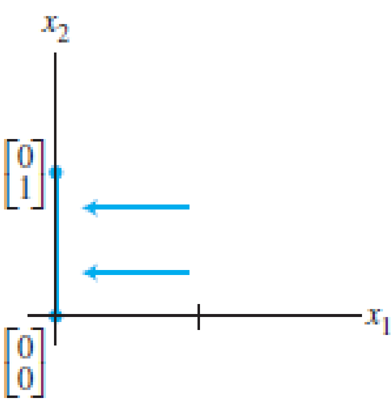
GEOMETRIC LINEAR TRANSFORMATIONS OF $\mathbb{R}^2$

TABLE 3 Shears

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

GEOMETRIC LINEAR TRANSFORMATIONS OF $\mathbb{R}^2$

TABLE 4 **Projections**

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

THE MATRIX OF A LINEAR TRANSFORMATION

- **Theorem 10**: Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a **unique matrix** A such that

$$T(x) = Ax \text{ for all x in } \mathbb{R}^n$$

- In fact, A is the m $\times$ n matrix whose j^{th} column is the vector $T(e_j)$, where e_j is the j^{th} column of the identity matrix in $\mathbb{R}^n$:

$$\boxed{A = [T(e_1), T(e_2), \ldots, T(e_n)]}$$

- **Proof**:

THE MATRIX OF A LINEAR TRANSFORMATION

- **Note 1:** The set of vectors $\{e_1, e_2, \dots, e_n\}$ in $\mathbb{R}^n$ is called a **standard basis of** $\mathbb{R}^n$.

- **Note 2:** The matrix $A = [T(e_1), T(e_2), \dots, T(e_n)]$ is called the **standard matrix for the linear transformation** T.

- **Note 3:** **Theorem 10** implies that **every linear transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as **a matrix transformation**, and vice versa.

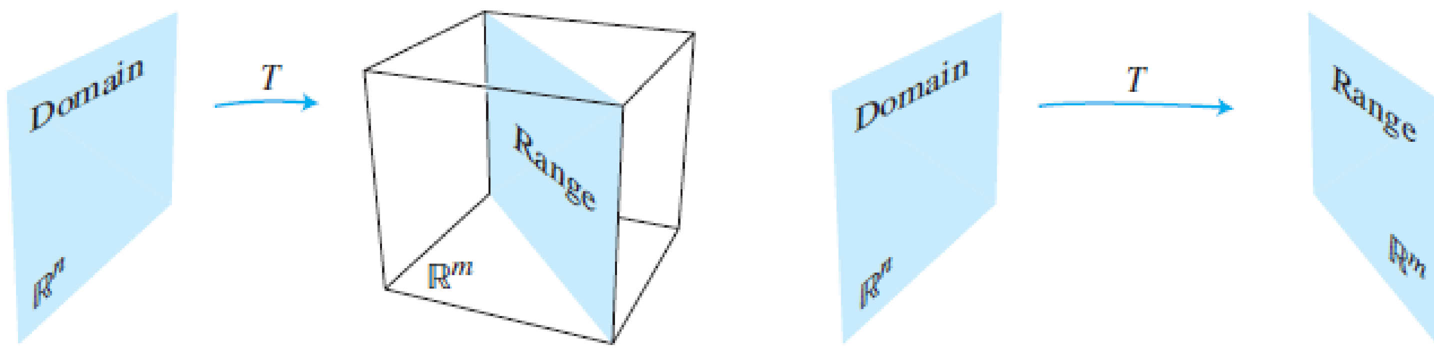
THE MATRIX OF A LINEAR TRANSFORMATION

- **Example** : Find the standard matrix A for the transformation $T(x)= 3x$, for x in $\mathbb{R}^2$.

- **Solution**:

ONTO MAPPINGS

- **Definition**: A mapping T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^n$ if *each* **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in $\mathbb{R}^n$. In other words the range of T is the whole codomain of T.



T is *not* onto $\mathbb{R}^m$ | T *is* onto $\mathbb{R}^m$

FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

ONTO MAPPINGS

When is a map onto?

- T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, for *each* **b** in $\mathbb{R}^m$, there exists a **solution** x in $\mathbb{R}^n$ of T(x)= b.

- Translating this problem into matrix notation we get

$$T(x) = Ax = b, \text{ where } A = [T(e_1), T(e_2), \dots, T(e_n)].$$

- When $T(x) = Ax = b$ has always a solution in stated in **Ch.1.4, Theorem 4.**

ONTO MAPPINGS

- **Theorem 4:** Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.
  a. For each **b** in $\mathbb{R}^m$, the equation $A\mathrm{x} = \mathrm{b}$ has a solution.
  b. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of A.
  c. The columns of A span $\mathbb{R}^m$.
  d. A has a pivot position in every **row.**

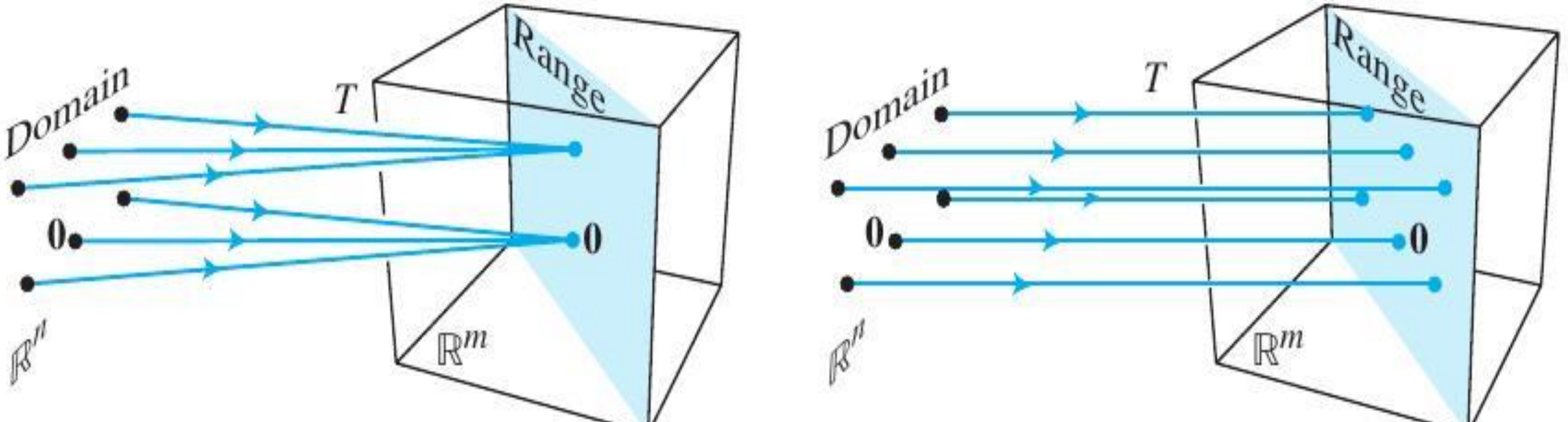
Hence we can conclude using just d.:

- **Theorem**: Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then T maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ if and only if A has a **pivot position** in every **row.**

ONE-TO-ONE MAPPINGS

Definition: A mapping T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$.

In other words for *each* **b** in the **range of *T*** there is *exactly one* **x** in $\mathbb{R}^n$ in such that $T(x) = b$.



T is *not* one-to-one   *T* is one-to-one

ONE-TO-ONE MAPPINGS

When is a map one-to-one?

- T is **one-to-one** if, for *<u>each</u>* **b** in $\mathbb{R}^m$, there exists **at most one solution** x in $\mathbb{R}^n$ of T(x)= b.

- Translating this problem into matrix notation we get

$$T(x) = Ax = b, \text{ where } A = [T(e_1), T(e_2), \ldots, T(e_n)].$$

- When $Ax = b$ has **at most one solution** in stated in **Ch.1.5, Theorem 6.**

ONE-TO-ONE MAPPINGS

- **Theorem 6:** Suppose the equation $A\mathrm{x} = \mathrm{b}$ is consistent for some given **b**, and let **p** be a solution. Then the solution set of $A\mathrm{x} = \mathrm{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$,where $\mathbf{v}_h$ is any solution of the homogeneous equation $Ax = 0$.

Hence if $Ax = b$ has a solution, then it has as many solutions as the equation $Ax = 0$. It follows:

- **Theorem**: Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then T is one-to-one if and only if the equation $T(x) = Ax = 0$ has **only the trivial solution**.

ONTO AND ONE-TO-ONE MAPPINGS - SUMMARY

- **Theorem 12**: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a) T maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ if and only if the **columns of *A* span** $\mathbb{R}^m$.

b) T is **one-to-one** if and only if the **columns of *A* are linearly independent**.

A more practical version is

- **Theorem**: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a) T maps $\mathbb{R}^n$ **onto** $\mathbb{R}^m$ if and only if A has **a pivot in every row**.

b) T is **one-to-one** if and only if the echelon form U of A has **a pivot in every column**.

ONTO AND ONE-TO-ONE MAPPINGS

- **Example**: Let T: $\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation, such that

$$T(e_1) = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, T(e_2) = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}, T(e_3) = \begin{bmatrix} 3 \\ -7 \\ -6 \end{bmatrix} \text{ and } T(e_4) = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

1.) Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$?

2.) Is *T* a one-to-one mapping?

3.) Find the x in $\mathbb{R}^4$, such that T(x) = 0.

Solution:

ONTO AND ONE-TO-ONE MAPPINGS