Math 22 –
Linear Algebra and its applications

- Lecture 7 -

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GENERAL INFORMATION

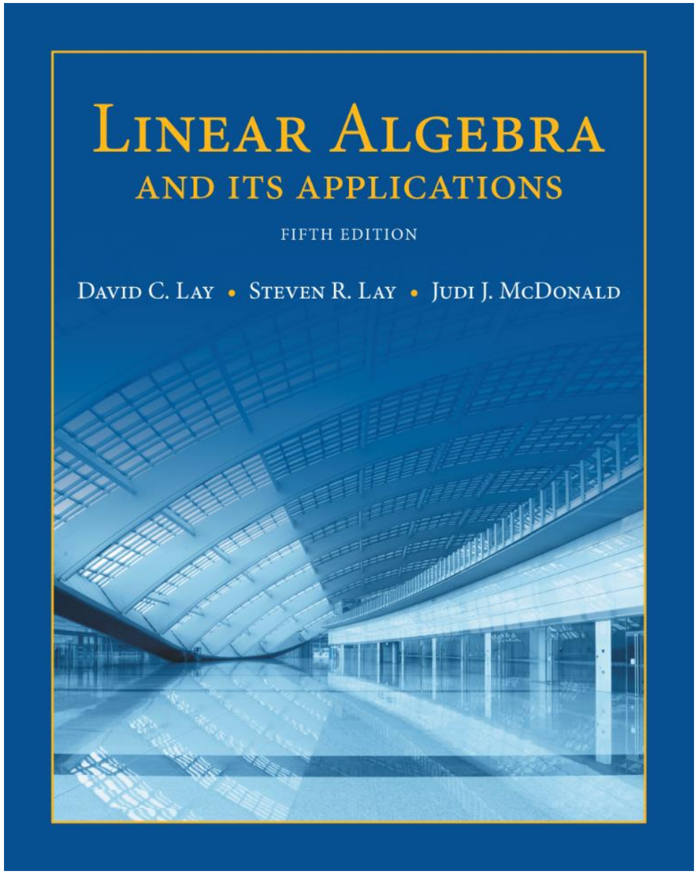
- **Office hours:** **Tu** 1-3 pm, Th, Sun 2-4 pm in KH 229
- **Tutorial:** **Tu**, Th, Sun 7-9 pm in KH 105
- **Homework:** Homework 2 due **Wednesday** at 4 pm in the boxes outside Kemeny 008. Separate your homework into **part A, part B, part C** and **part D** and staple it.
- **Midterm 1**: **Monday** Oct 7 from **4-6 pm** in **Carpenter 013**

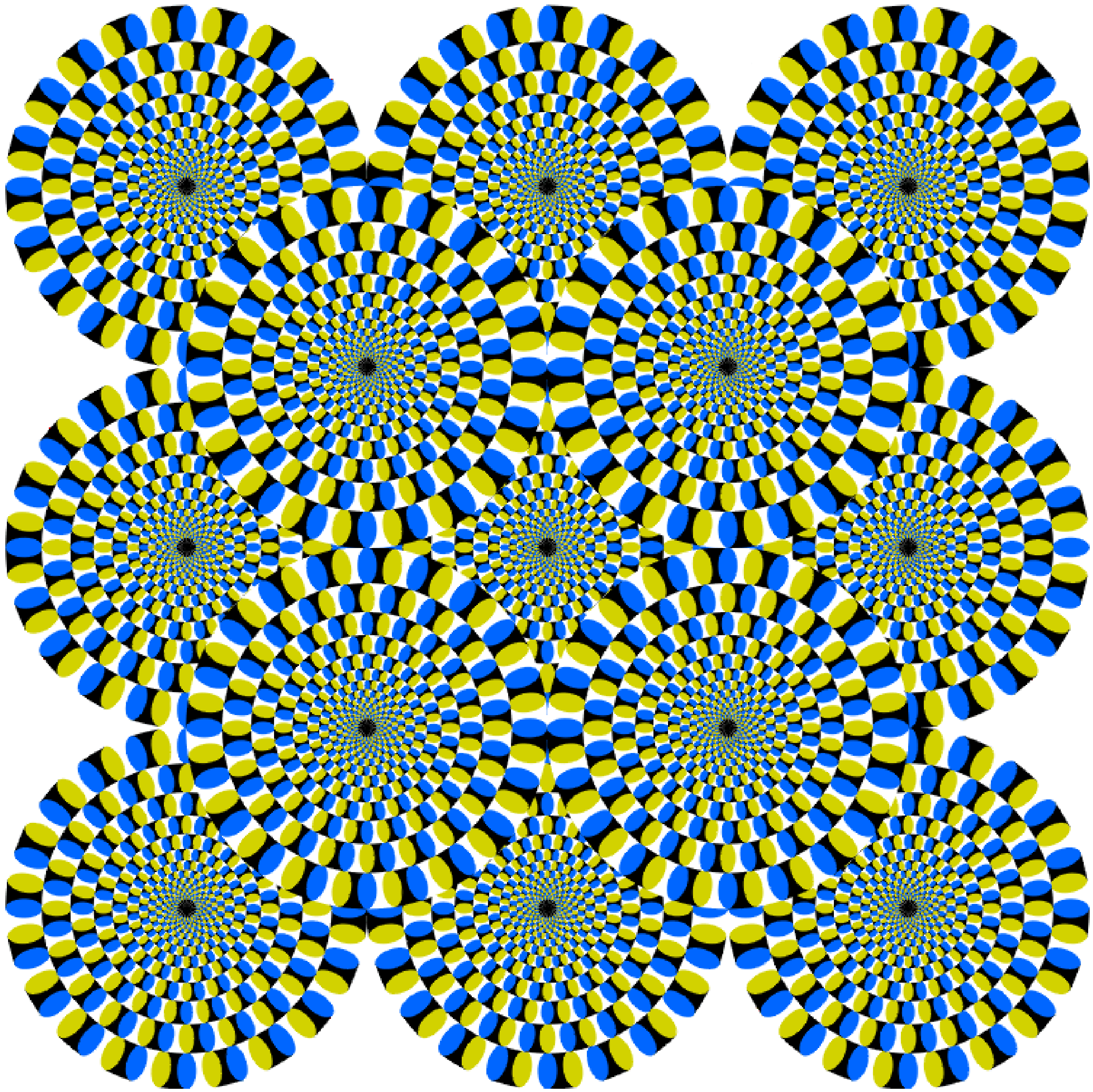
  Topics: till this **Thursday** (included)

1 Linear Equations in Linear Algebra

1.8

INTRODUCTION TO LINEAR TRANSFORMATIONS





LINEAR TRANSFORMATIONS

- **Summary:** If a transformation T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is **linear**, then it maps Euclidean subspaces to Euclidean subspaces.

GEOMETRIC INTERPRETATION

- **Example:**

GEOMETRIC INTERPRETATION

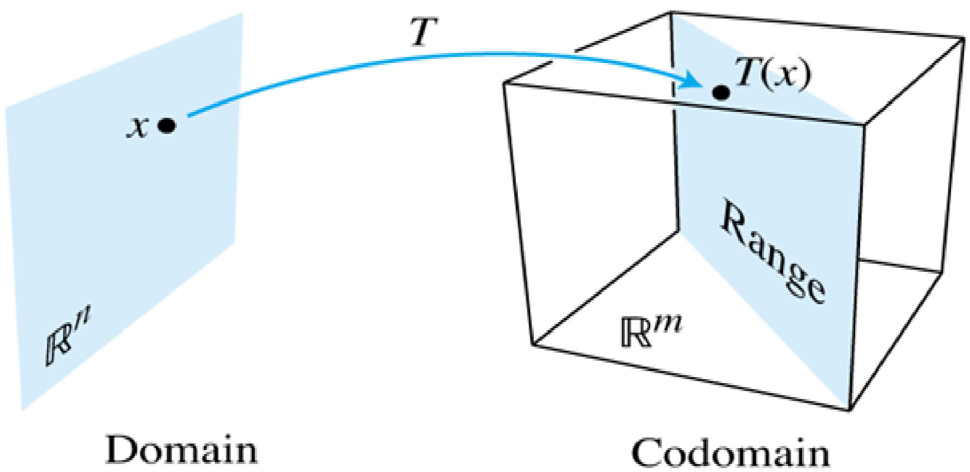
TRANSFORMATIONS

- **Definition:** A **transformation** (or **function** or **mapping**) T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(\mathbf{x})$ in $\mathbf{R}^m$.
- The set $\mathbf{R}^n$ is called **domain** of T, and $\mathbf{R}^m$ is called the **codomain** of T.
- We use the notation

$$T: \mathbf{R}^n \rightarrow \mathbf{R}^m .$$

TRANSFORMATIONS

- **Definition:** Let T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ be a transformation or mapping.
- For $\mathbf{x}$ in $\mathbf{R}^n$, the vector $T(\mathbf{x})$ in $\mathbf{R}^m$ is called the **image** of $\mathbf{x}$ (under the action of T).
- The set of all images $T(\mathbf{x})$ is called the **range** of T.



Domain, codomain, and range of $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$.

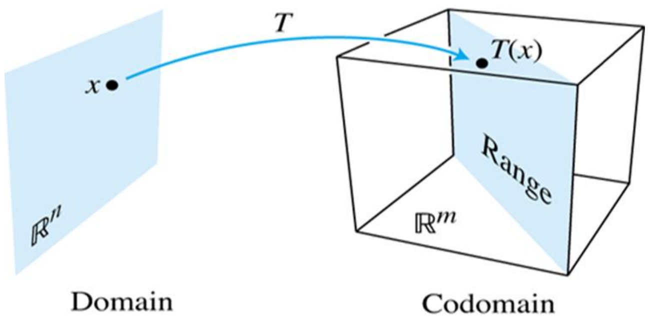
MATRIX TRANSFORMATIONS

- **Definition:** Let A be an $m \times n$ matrix. A **matrix transformation** is the associated map given by ***T*(x) = Ax**, for any x in $\mathbf{R}^n$

- For simplicity, we denote such a matrix transformation often by

$$x \mapsto Ax.$$

- As each column of A has m columns, the domain of T is $\mathbf{R}^n$ and the codomain of T is $\mathbf{R}^m$.



Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

LINEAR TRANSFORMATIONS

- **Definition:** A transformation (or mapping) T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is **linear** if:
  i. $T(\mathrm{u}+\mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$ for all **u**, **v** in $\mathbf{R}^n$.
  ii. $T(c\mathrm{u}) = cT(\mathrm{u})$ for all c in **R** and all **u** in $\mathbf{R}^n$.

A matrix transformation A: $\mathbf{R}^n \rightarrow \mathbf{R}^m$, $\mathrm{x} \mapsto A\mathrm{x}$ is **always linear** as for all **u**, **v** in $\mathbf{R}^n$ and c in **R:**

a) A(u+v) = Au + Av

b) A(cu) = cAu

LINEAR TRANSFORMATIONS

- **Consequences:**
- As $T(\mathrm{u}+\mathrm{v})=T(\mathrm{u})+T(\mathrm{v})$ and $T(c\mathrm{u})=cT(\mathrm{u})$ we have:

  iii. $T(0) = 0$

  iv. $T(c_1\mathrm{v}_1+\ldots+c_p\mathrm{v}_p)=c_1T(\mathrm{v}_1)+\ldots+c_pT(\mathrm{v}_p)$

- If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbf{R}^n$ are vectors then *iv.* implies that

$$T(\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}) = \mathrm{Span}\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$$

Proof:

LINEAR TRANSFORMATIONS

- **Consequences:**

  iv. $T(c_1\text{v}_1 + \ldots + c_p\text{v}_p) = c_1T(\text{v}_1) + \ldots + c_pT(\text{v}_p)$

- In engineering and physics, iv. is referred to as a ***superposition principle*.**
- Think of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ as signals that go into a system and $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)$ as the responses of that system to the signals.

- The system satisfies the superposition principle if whenever an input is expressed as a linear combination of such signals, the system's response is the *same* linear combination of the responses to the individual signals.

MATRIX TRANSFORMATIONS

- **Example:** Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathrm{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ and $\mathrm{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

  and define a transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ by $T(\mathrm{x}) = A\mathrm{x}$.

  a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

  b. Find an $\mathbf{x}$ in $\mathbf{R}^n$ whose image under T is $\mathbf{b}$.

  c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

  d. Determine if $\mathbf{c}$ is in the range of the transformation T.

MATRIX TRANSFORMATIONS

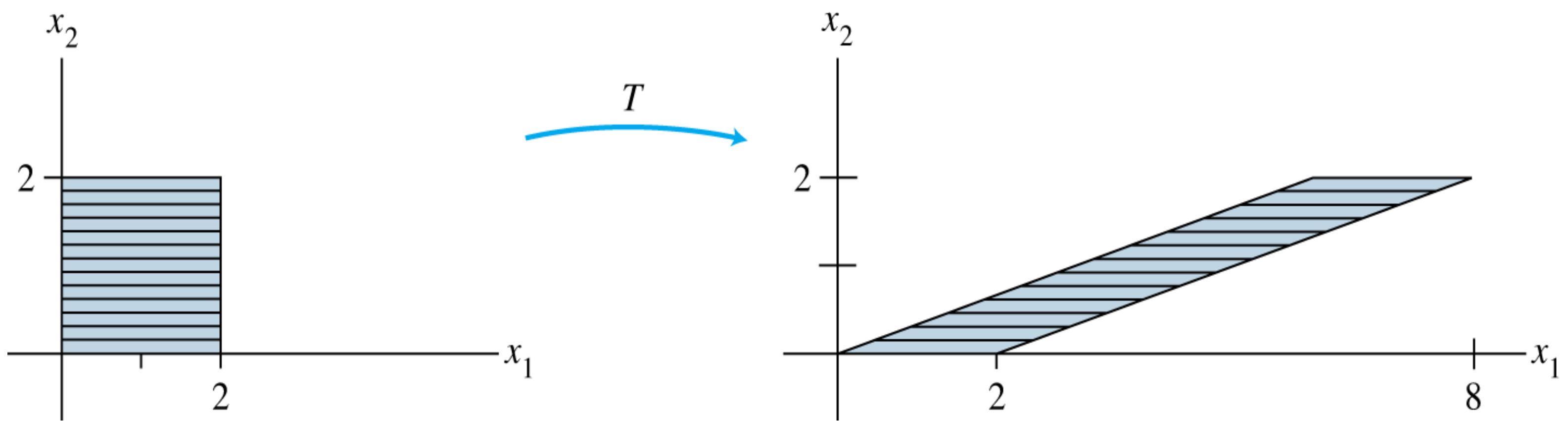
- **Solution:**

MATRIX TRANSFORMATIONS

SHEAR TRANSFORMATION

- **Example 2:** Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation T: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $T(\mathrm{x}) = A\mathrm{x}$ is called a **shear transformation**.

- The image of the square below is a parallelogram:



SCALAR MULTIPLICATION

- **Example:** Given a scalar r, define $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by

$$T(\mathrm{x}) = r\mathrm{x}.$$

- T is called a **contraction** when $0 \le r \le 1$ and a **dilation** when $r > 1$.

ROTATION

- **Example:** Let T: $\mathbf{R}^2 \to \mathbf{R}^2$ be given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- T is called a **rotation with angle t**.