Math 22 –
Linear Algebra and its applications

- Lecture 6 -

Instructor: Bjoern Muetzel

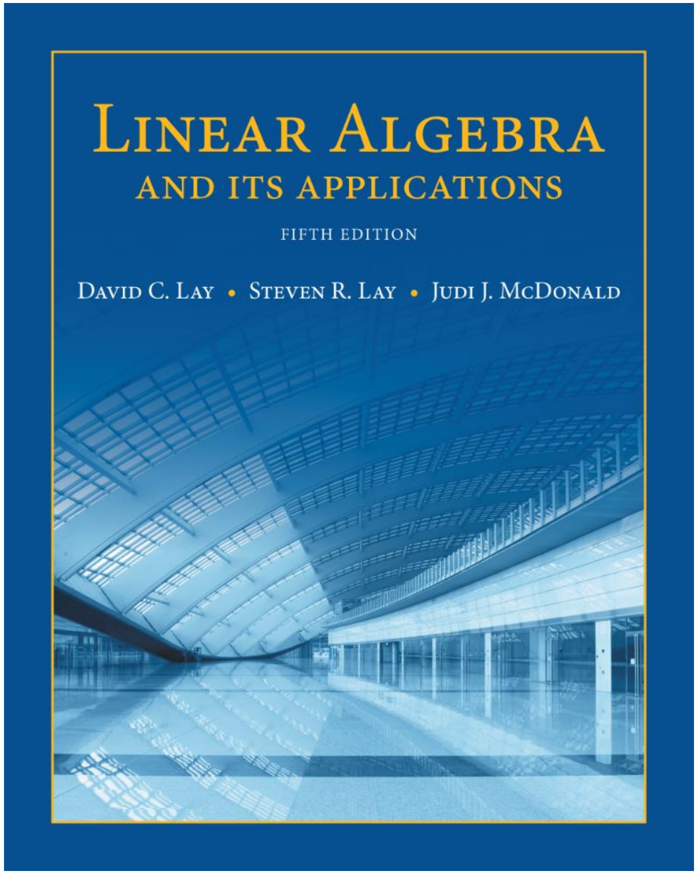
GENERAL INFORMATION

- **<u>Office hours:</u>** Tu 1-3 pm, Th, **Sun** 2-4 pm in KH 229
- **<u>Tutorial:</u>** Tu, Th, **Sun** 7-9 pm in KH 105
- **<u>Homework:</u>** <u>Homework 2</u> due **next Wednesday** at 4 pm in the boxes outside Kemeny 008. Separate your homework into **part A, part B** and **part C** and **D** and staple it.

1 Linear Equations in Linear Algebra

1.7

LINEAR INDEPENDENCE



LINEAR INDEPENDENCE

- **<u>Summary:</u>** Span{$\mathbf{v}_1, \ldots, \mathbf{v}_p$} might be generated by **less than** p vectors. In this case the list $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is **redundant**.
- This is equal to saying that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are **linearly dependent**.

LINEAR INDEPENDENCE

- **Definition:** An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if the vector equation

$$\boxed{x_1\mathrm{v}_1 + x_2\mathrm{v}_2 + \ldots + x_p\mathrm{v}_p = 0}$$

has **only the trivial solution $\mathbf{x_1 = x_2 = \ldots = x_p = 0}$**.

Otherwise the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent.** This means that there exist **weights** $c_1, \ldots, c_p$, **not all zero**, such that

$$\boxed{c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \ldots + c_p\mathrm{v}_p = 0} \quad (1)$$

Equation (1) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$ when the weights are not all zero.

GEOMETRIC INTERPRETATION

- Example:

GEOMETRIC INTERPRETATION

LINEAR INDEPENDENCE IN MATRIX NOTATION

- Suppose that we begin with a matrix $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$ instead of a set of vectors.

- The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{0}.$$

- The columns of A are **linearly dependent**, if and only if $A\mathbf{x} = \mathbf{0}$ has **nontrivial solutions**.

- Thus, the columns of matrix A are **linearly independent** if and only if the equation $A\mathbf{x} = \mathbf{0}$ has **only the trivial solution**.

LINEAR INDEPENDENCE

- **Example 1:** Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

    a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

    b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

LINEAR INDEPENDENCE

THE ZERO VECTOR

- **Theorem:** If a set $S = \{v_1, \ldots, v_p\}$ in $\mathbf{R}^n$ contains the zero vector, then the set is linearly dependent.

- **Proof:**

  By renumbering the vectors, we may suppose that $v_1 = 0$.

  Then the equation shows that S is linearly dependent.

SETS OF ONE VECTOR

- A set containing only one vector – say, **v** – is linearly independent if and only if **v** is not the zero vector.

SETS OF TWO VECTORS

- A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.

- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

SETS OF TWO OR MORE VECTORS

- **Theorem 7: (Characterization of Linearly Dependent Sets)**
  An indexed set $S = \{v_1, ..., v_p\}$ of two or more vectors is **linearly dependent** if and only if **at least one** of the vectors in S is a **linear combination** of the others.

- This is equal to saying that there exists some $\mathbf{v}_j$ in $\mathbf{v}_1, \ldots, \mathbf{v}_p$ such that

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_p\}$$

SETS OF TWO OR MORE VECTORS

- **Proof:**

SETS OF TWO OR MORE VECTORS

- **Note 1: Theorem 7** does ***NOT*** imply that EVERY vector in a linearly dependent set is a linear combination of the preceding vectors.
- **Note 2:** A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

- **Example for Note 2:** For $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

  $\{v_1, v_2, v_3\}$ are linearly dependent, but v_3 can not be written as a linear combination of the other two vectors.

SETS OF TWO OR MORE VECTORS

- **Theorem 8:** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is linearly dependent if $p > n$.

- **Proof:** Let $A = \begin{bmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_p \end{bmatrix}$.
- Then A is $n \times p$, and the equation $A\mathrm{x} = 0$ corresponds to a system of n equations in p unknowns.
- If $p > n$, there are more variables than equations, so there must be a _______ variable.
- Hence $A\mathrm{x} = 0$ has a nontrivial solution, and the columns of A are linearly dependent.

SETS OF TWO OR MORE VECTORS

- **Note:** **Theorem 8** says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

SETS OF TWO OR MORE VECTORS

- **Example:** Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by **u** and **v**, and explain why a vector **w** is in Span {**u**, **v**} if and only if {**u**, **v**, **w**} is linearly dependent.

SETS OF TWO OR MORE VECTORS

- **Solution:**