Math 22 –
Linear Algebra and its applications

- Lecture 4 -

Instructor: Bjoern Muetzel

GENERAL INFORMATION

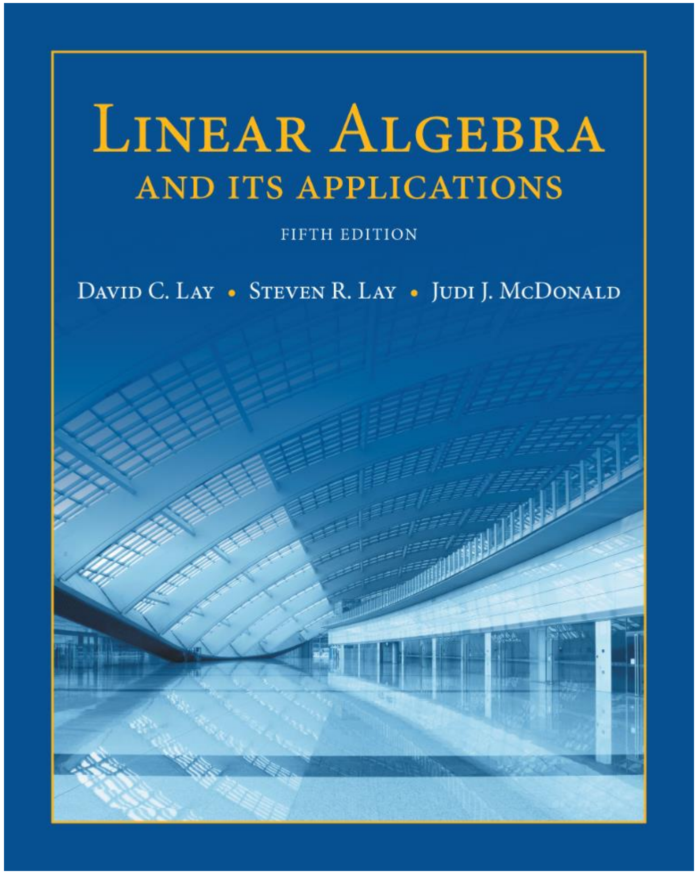
- **Office hours:** Tu 1-3 pm, **Th**, Su 2-4 pm in KH 229
- **Tutorial:** Tu, **Th**, Sun 7-9 pm in KH 105
- **Homework:** Homework 1 due **today** at 4 pm in the boxes outside Kemeny 008. Separate your homework into **part A, part B** and **part C** and staple it.
- **Attention:** This **Thursday** the **x-hour** will be a **lecture**:

  Section 1: 12:15 - 1:05 pm in Kemeny 007
  Section 2: 1:20 - 2:10 pm in Kemeny 007

1 Linear Equations in Linear Algebra

1.4

THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$ AND SOLUTION SETS OF LINEAR EQUATIONS



Aims:

1.) Find out which **properties** a **coefficient matrix A** of a system of linear equations with **augmented matrix [A|b]** must have, such that the system has a **solution for any vector b**.

2.) **Refinement / Reinterpretation** of the description of the **solution set** of a system of linear equations **in terms of vectors**.

GEOMETRIC INTERPRETATION

- **Example:**

GEOMETRIC INTERPRETATION

MATRIX VECTOR MULTIPLICATION

- **Definition:** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the **product of** A **and x**, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries of $\mathbf{x}$ as weights; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n .$$

- **Note:** $A\mathbf{x}$ is defined only if the **number of columns** of A **equals** the **number of entries** in $\mathbf{x}$.

- **Theorem:** If A is an $m \times n$ matrix, **u** and **v** are vectors in $\mathbf{R}^n$ and c is a number, then

  a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

  b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

  Proof:

MATRIX VECTOR MULTIPLICATION

Example: Compute $A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

- **Solution:**

MATRIX VECTOR MULTIPLICATION

- Now, write a system of linear equations as a vector equation involving a linear combination of vectors.
- **Example:** The following system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \qquad (1)$$

is equivalent to

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **<u>Theorem 3:</u>** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if b is in $\mathbb{R}^n$, then the matrix equation

$$\boxed{A\mathbf{x} = \mathbf{b}}$$

has the same solution set as the vector equation

$$\boxed{x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}}$$

which, in turn, has the same solution set as the system of linear equations whose **augmented matrix** is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

MATRIX EQUATION $Ax = \mathrm{b}$

- **Definition:** The matrix with 1s on the diagonal and 0s elsewhere is called an **identity matrix** and is denoted by I.

  Example:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Definition:** We say that the columns of the $m \times p$ matrix $A = [\mathbf{a}_1, \ldots, \mathbf{a}_p]$, span $\mathbf{R}^m$, if every vector **b in** $\mathbf{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_p$, i.e.

$$\boxed{\mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_p\} = \mathbf{R}^m}.$$

EXISTENCE OF SOLUTIONS

- The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if **b** is a linear combination of the columns of *A*.

- **Theorem 4:** Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

  a. For each **b** in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
  b. Each **b** in $\mathbf{R}^m$ is a linear combination of the columns of *A*.
  c. The columns of *A* span $\mathbf{R}^m$.
  d. *A* has a pivot position **in every row**.

Proof of Theorem 4:

- Statements (a), (b), and (c) are logically equivalent.

  So, it suffices to show that (a) and (d) are either both true or false.

  Idea: Look at the **echelon form** of A.

  Let *U* be an echelon form of a matrix *A*. Given **b** in $\mathbf{R}^m$, we can row reduce the augmented matrix [A|**b**] to an augmented matrix [U|**d**] for some **d** in $\mathbf{R}^m$:

- **(d) implies (a):**

  If statement (d) is true, then each row of *U* contains a pivot position, which asserts that the linear system corresponding to [A|**b**] and [U|**d**] has a solution, independent of **b** or **d**.

  So [A|**b**] has a solution for any **b**, and (a) is true. So (d) implies (a).

PROOF OF THEOREM 4

- **(a) implies (d):** This is equivalent to the statement: If **(d) is false**, **then (a) is false**.
- If (d) is false then the last row of *U* is all zeros.

  Let **d** be any vector with a 1 in its last entry.

  Then $[U|\mathbf{d}]$ has no solution.
- Since row operations are reversible, $[U|\mathbf{d}]$ can be transformed into the form $[A|\mathbf{b}]$.
- The new system $A\mathbf{x} = \mathbf{b}$ has also no solution, and (a) is false.

HOMOGENEOUS LINEAR SYSTEMS

Aim: Refinement / Reinterpretation of the parametric description of the solution set of a system of linear equations.

HOMOGENEOUS LINEAR SYSTEMS

- A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and 0 is the zero vector in $\mathbf{R}^m$

- Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution,

  namely $\mathbf{x} = \mathbf{0}$ in $\mathbf{R}^n$

- This zero solution is usually called the **trivial solution**.

- The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a **nontrivial solution** if and only if the equation has at least one free variable.

- A system of linear equations is said to be **nonhomogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{b}$, where **b not 0**.

HOMOGENEOUS LINEAR SYSTEMS

- **Example:** Determine the solution of the

  1.) homogeneous system $A\mathbf{x} = \mathbf{0}$

  2.) nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \text{ and } \mathrm{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}.$$

- Does the homogeneous system have a nontrivial solution?

HOMOGENEOUS LINEAR SYSTEMS

HOMOGENEOUS LINEAR SYSTEMS

PARAMETRIC VECTOR FORM

- The equation $\mathbf{x} = t\mathbf{v}$ (with t in **R,** v in **R**^{m}), is a **parametric vector equation** of a **line**.

- The equation of the form $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (s, t in **R,** u,v in **R**^{m}) is called a **parametric vector equation** of a **plane**.

- Whenever a solution set is described explicitly with vectors, we say that the solution is in **parametric vector form**.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- **Theorem 6:** Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given **b**, and let **p** be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

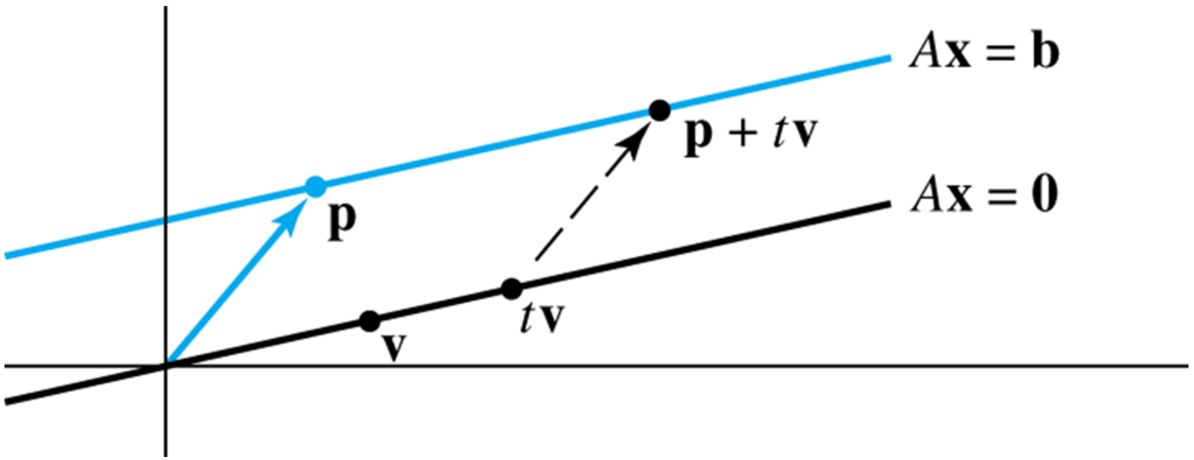
$$\mathbf{w} = \mathbf{p} + \mathbf{v}_h$$

where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Proof:

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- **Note:** This theorem says that if $Ax = b$ has a solution, then the solution set is obtained by translating the solution set of $Ax = 0$, using any particular solution **p** of $Ax = b$ for the translation.



WRITING A SOLUTION SET IN PARAMETRIC VECTOR FORM

1. **Row reduce** the augmented matrix to reduced echelon form.
2. Express each **basic variable in terms of** any **free variables** appearing in an equation.
3. Express each **free variable** by **itself.**
4. **Decompose** the general solution **x** into a **linear combination of vectors** (with numeric entries) using the **free variables** as **parameters**.
5. **Replace** the free variables by simple letters / parameters.