# Math 22 –
# Linear Algebra and its applications

# - Lecture 3 -

**Instructor:** Bjoern Muetzel

# GENERAL INFORMATION

- **Office hours: Tu 1-3 pm**, Th, Su 2-4 pm in KH 229

- **Tutorial: Tu**, Th, Sun 7-9 pm in KH 105

- **Homework:** Homework 1 due **this Wednesday** at 4 pm in the boxes outside Kemeny 008. Separate your homework into **part A, part B** and **part C** and staple it.

- **Attention:** This **Thursday** the **x-hour** will be a **lecture**:

  **Section 1:** 12:15 - 1:05 pm in Kemeny 007
  **Section 2:** 1:20 - 2:10 pm in Kemeny 007

# REVIEW - ROW REDUCTION ALGORITHM

## FORWARD PHASE

- **STEP 1:** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
- **STEP 2:** Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
- **STEP 3:** Use row replacement operations to create zeros in all positions below the pivot.
- **STEP 4:** Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

## BACKWARD PHASE

- **STEP 5:** Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

# REVIEW – SOLVING A SYSTEM OF LINEAR EQUATIONS

**Example:**

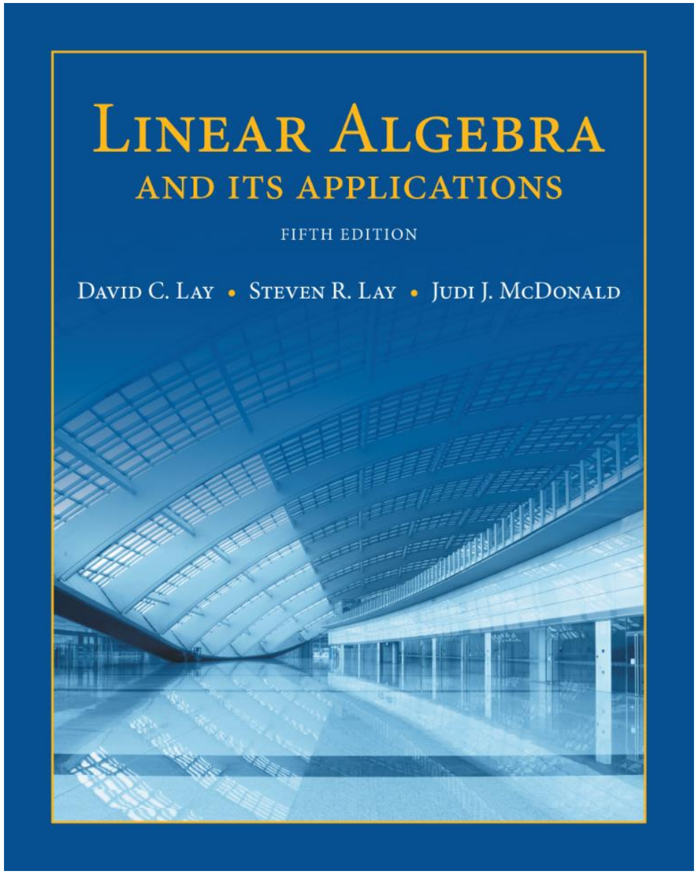
# ECHELON FORM (EF) AND REDUCED ECHELON FORM (REF / RREF)

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# 1 Linear Equations in Linear Algebra

## 1.3

## VECTOR EQUATIONS



# GEOMETRIC INTERPRETATION

# GEOMETRIC INTERPRETATION

# VECTORS

- A matrix with only one column is called a **column vector**, or simply a **vector**.

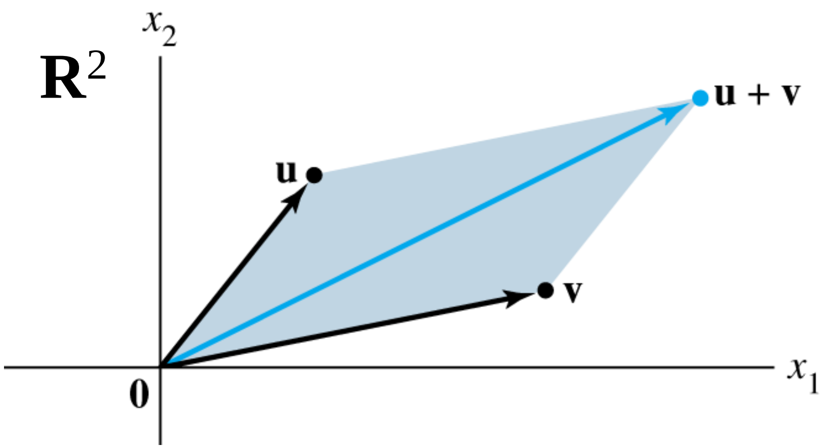
$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

- The set of all vectors with n entries is denoted by $\mathbf{R}^n$
- The **R** stands for the **real numbers** that appear as entries in the vector
- Two vectors in $\mathbf{R}^n$ are equal if and only if their corresponding entries are equal.

# VECTORS

- Given two vectors **u** and **v** in $\mathbf{R}^n$, their **sum** is the vector **u + v** obtained by adding corresponding entries of **u** and **v**.

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$




- Given a vector **u** and a real number *c*, the **scalar multiple** of **u** by *c* is the vector *c***u** obtained by multiplying each entry in **u** by *c*.

# VECTORS

- **Examples:**

# ALGEBRAIC PROPERTIES OF $\mathbf{R}^n$

- The vector whose entries are all zero is called the **zero vector** and is denoted by **0**.
- As all **algebraic operations** are performed **componentwise**, we have:
- For all **u**, **v**, **w** in $\mathbf{R}^n$ and all scalars $c$ and $d$ in **R**:

(i) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(ii) $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$

(iii) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$, where $-\mathbf{u} = (-1)\cdot\mathbf{u}$

(iv) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(v) $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vi) $(\mathbf{u}+\mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v}+\mathbf{w})$

(vii) $1\cdot\mathbf{u} = \mathbf{u}$

# LINEAR COMBINATIONS

- **Definition:** Given vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ in $\mathbf{R}^n$ and given scalars $c_1, c_2, ..., c_p$, the vector **y** defined by

$$\boxed{\mathbf{y} = c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p}$$

  is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.

- **Note:** The **weights** in a linear combination can be any real numbers, including zero.

- **Example:**

# LINEAR COMBINATIONS

- A vector equation, where the ($\mathbf{a}_\mathbf{i}$ ) are vectors, of the form

$$x_1\mathrm{a}_1 + x_2\mathrm{a}_2 + \ldots + x_n\mathrm{a}_n = \mathrm{b}$$

  has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \cdots & \mathrm{a}_n & \mathrm{b} \end{bmatrix} \quad (5)$$

- In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

# LINEAR COMBINATIONS

- **Example :** Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

  Determine whether **b** can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. If so, what are the weights?

# LINEAR COMBINATIONS

# LINEAR COMBINATIONS

- **Definition:** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbf{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by

$$\text{Span}\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_p\}$$

and is called the **subset of $\mathbf{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

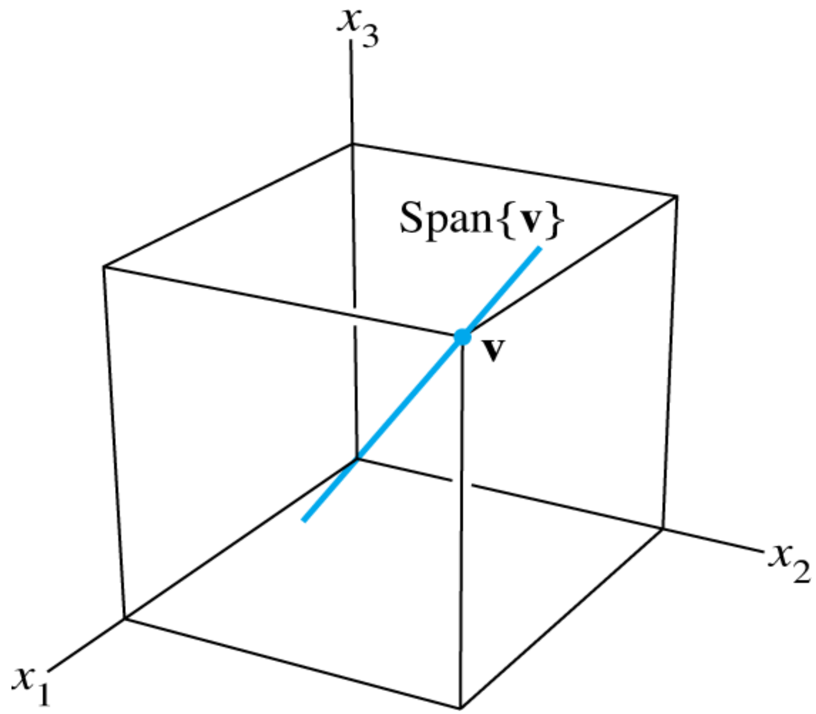
That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \ldots + c_p\mathrm{v}_p$$

with $c_1, \ldots, c_p$ scalars.

# GEOMETRIC INTERPRETATION

- Let **v** be a nonzero vector in $\mathbf{R}^n$. Then Span {**v**} is the set of all scalar multiples of **v**, which is the set of points on the line in $\mathbf{R}^n$ through **v** and **0**.
- **Example:** **v** in $\mathbf{R}^3$



# GEOMETRIC INTERPRETATION

- If **u** and **v** are nonzero vectors in $\mathbf{R}^n$ , with **v** not a multiple of **u**, then Span {**u, v**} is the plane in $\mathbf{R}^n$ that contains **u**, **v**, and **0**.
- In particular, Span {**u**, **v**} contains the line in $\mathbf{R}^n$ through **u** and **0** and the line through **v** and **0**.
- **Example:** **u** , **v** in $\mathbf{R}^3$