Math 22 –
Linear Algebra and its applications

- Lecture 2 -

Instructor: Bjoern Muetzel

GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, Th, Su 2-4 pm in KH 229

- **Tutorial:** Tu, Th, Sun 7-9 pm in KH 105

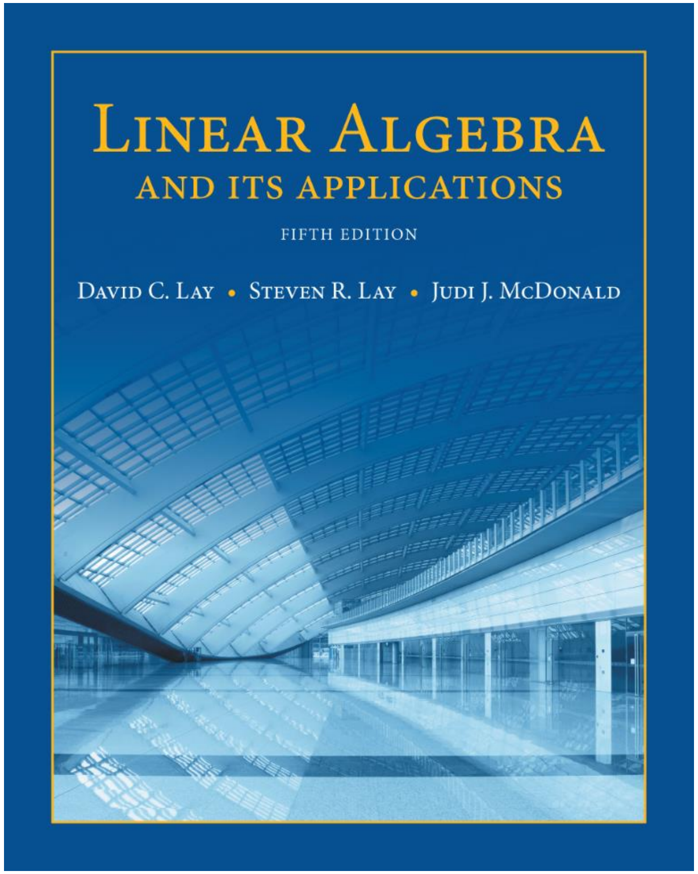
- **Homework:** Homework 1 due Wednesday Sept 25 before class. Total homework makes up 20% of your grade.

- **Practice problems:** Do the suggested practice problems after each class. If you have difficulties come to the office hours or the tutorial.

1 Linear Equations in Linear Algebra

1.2

Row Reduction and Echelon Forms



ECHELON FORM

Aim: Learn an algorithm that can solve any system of linear equations or state that it has no solution.

ECHELON FORM

<u>Aim:</u> To solve a system of linear equations we have to bring the corresponding augmented matrix into echelon form using elementary row operations.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Summary:

ECHELON FORM

- A matrix is in **(row) echelon form** if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading nonzero entry of a row is in a column to the right of the leading nonzero entry of the row above it.
  3. All entries in a column below a leading nonzero entry are zeros.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ECHELON FORM

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** or **reduced row echelon form (RREF):**

    4. The leading entry in each nonzero row is 1.
    5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ECHELON FORM

- A **pivot position** in a matrix *A* is a location in *A* that corresponds to a **leading 1** in the **reduced echelon form of *A***. The pivot positions are the **same** in the **unreduced echelon** form.

- A **pivot column** is a column of *A* that contains a pivot position.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ECHELON FORM

- Any nonzero matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations.

- However, the reduced echelon form one obtains from a matrix is unique.

Theorem 1: (Uniqueness of the Reduced Echelon Form)
Each matrix is row equivalent to one and only one reduced echelon matrix.

ECHELON FORM

- The **echelon form** of the matrix tells us whether the system has **0,1 or** ∞ **many** solutions.

- The **row reduced echelon form** provides a **parametric description** of the solution set, if a solution exists.

ECHELON FORM

Examples:

ROW REDUCTION ALGORITHM

FORWARD PHASE

- **STEP 1:** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
- **STEP 2:** Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
- **STEP 3:** Use row replacement operations to create zeros in all positions below the pivot.
- **STEP 4:** Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

BACKWARD PHASE

- **STEP 5:** Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

ROW REDUCTION ALGORITHM

- **Example:** Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

ROW REDUCTION ALGORITHM

- This is the reduced echelon form of the original matrix.

- Note that we have to apply **at most** the **number of rows** times the **STEPS 1-3** to obtain this result.

SOLUTIONS OF LINEAR SYSTEMS

- **Example:**

SOLUTIONS OF LINEAR SYSTEMS

- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

- The variables that correspond to pivot columns in the matrix are called **basic variables**. The other variables are called a **free variable**.

- Whenever a system is consistent then the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables.

EXISTENCE AND UNIQUENESS THEOREM

Theorem 2: (Existence and Uniqueness Theorem)

A linear system is consistent if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \ldots \ 0 \ \ b] \text{ with } b \text{ nonzero.}$$

- If a linear system is consistent, then the solution set contains either
- (i) a unique solution, when there are no free variables, or
- (ii) infinitely many solutions, when there is at least on free variable.

ROW REDUCTION TO SOLVE A LINEAR SYSTEM

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.