Math 22 –
Linear Algebra and its applications

- Lecture 27 -

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GENERAL INFORMATION

- **Office hours:** **Tu 1-3 pm**, Th, Sun 2-4 **pm** in **KH 229**

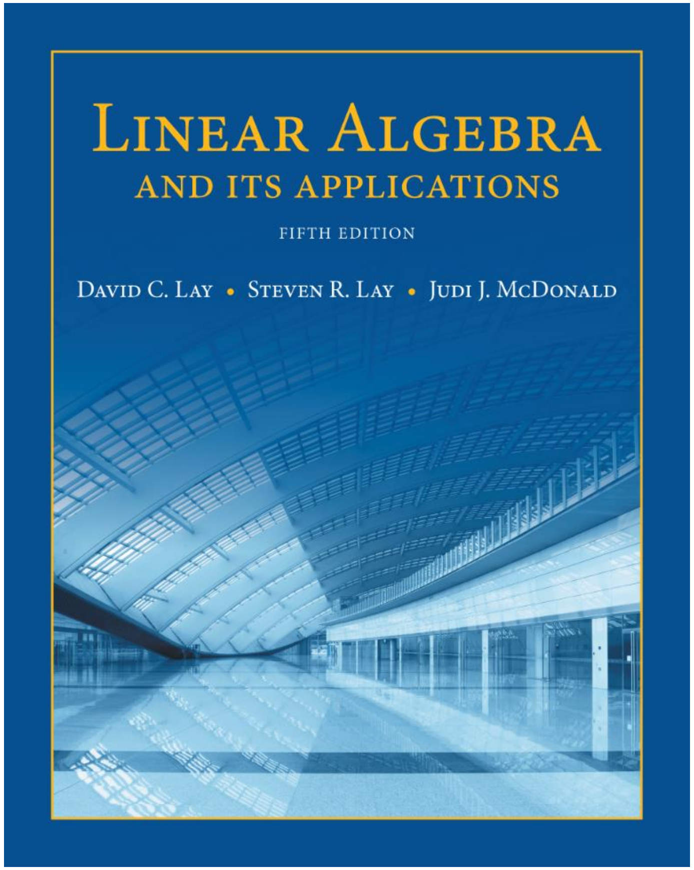
  Tutorial: **Tu**, Th, Sun **7-9 pm** in **KH 105**

- **Homework 8**: due **Wednesday** at **4 pm** outside **KH 008.** Please give in **part B, C** and **D.** There is **no part A.**

5 Eigenvalues and Eigenvectors

5.3

DIAGONALIZATION



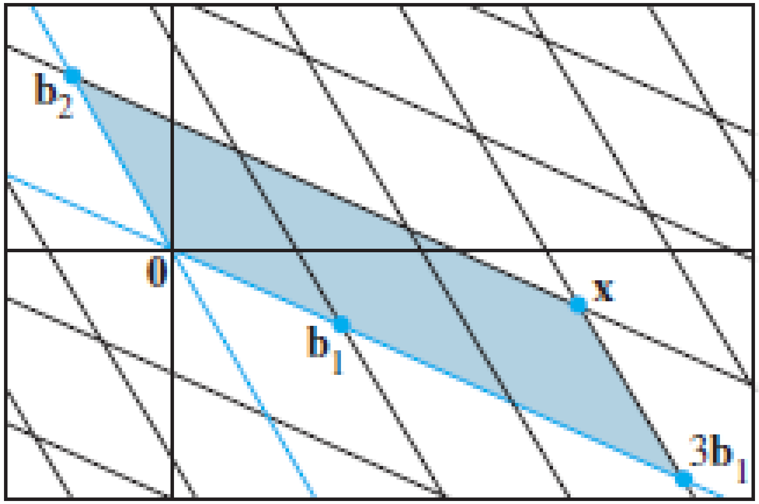
Summary:

Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, then there is **always a good basis** on which the transformation has a very simple form. In the best case there is a **basis of eigenvectors** and the matrix is **diagonal** with respect to this basis.

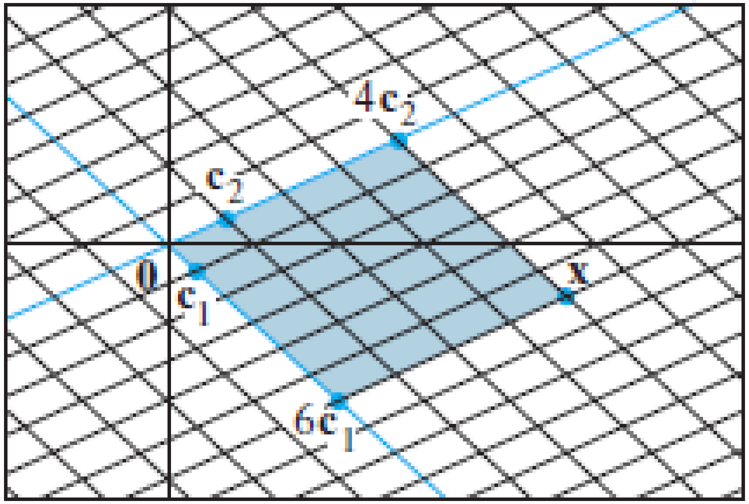
GEOMETRIC INTERPRETATION

Example: Consider the two bases for $\mathbb{R}^2$

$$B = \{b_1, b_2\} = \left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\} \text{ and } C = \{c_1, c_2\} = \left\{\begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}, \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}\right\}$$



(a)



(b)

Let $T: \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto T(x) = Ax$,where $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

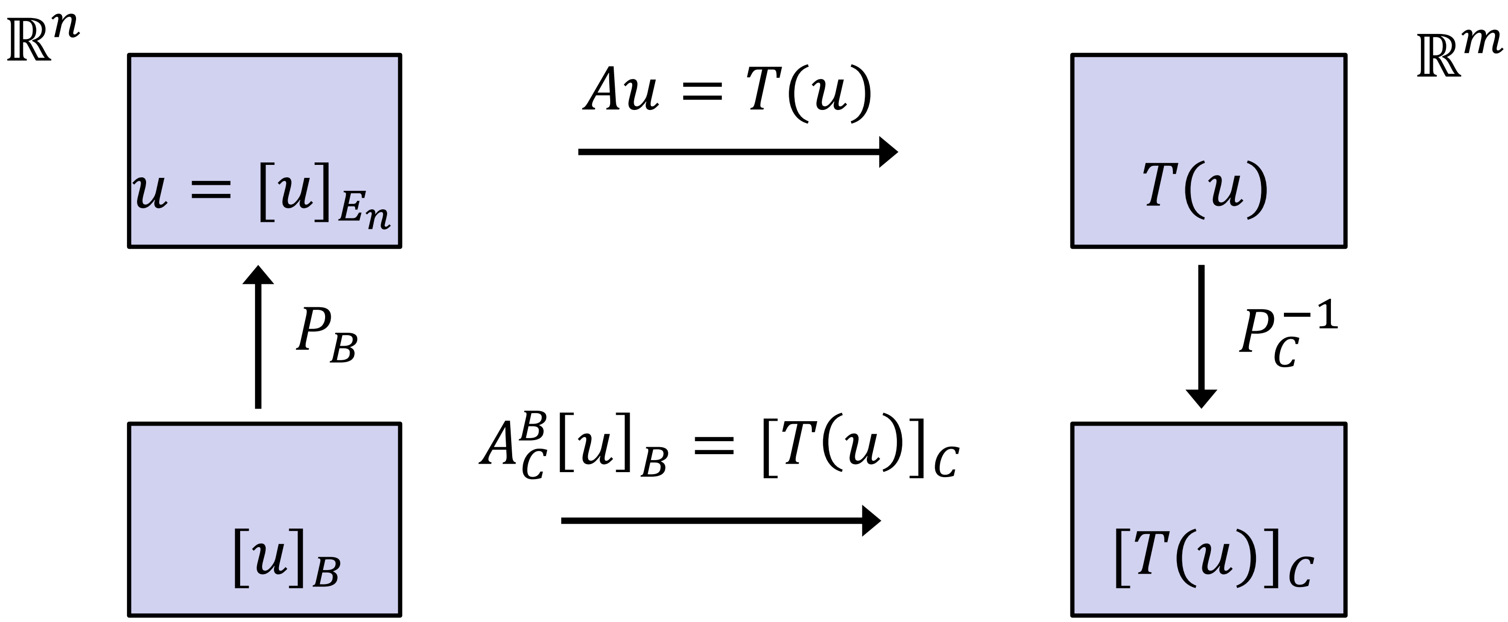
<u>**Question:**</u> **What is the matrix of *T* with respect to the bases B and C?**

LINEAR TRANSFORMATION WITH RESPECT TO DIFFERENT BASES

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Let E_n in $\mathbb{R}^n$ and E_m in $\mathbb{R}^m$ be the **standard bases**.

Given **different** bases **B= {b$_1$, …, b$_n$}** of $\mathbb{R}^n$ and **C= {c$_1$, …, c$_m$}** of $\mathbb{R}^m$.

What is the matrix of the linear transformation *T* with respect to the bases B and C ?



More precisely: What is the matrix $\boldsymbol{A_C^B}$, such that

$$\boxed{A_C^B[u]_B = [T(u)]_C} \text{ for all } u \text{ in } \mathbb{R}^n$$

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Given bases **B= {b$_1$, …, b$_n$}** of $\mathbb{R}^n$ and **C= {c$_1$, …, c$_m$}** of $\mathbb{R}^m$. Let A_C^B be the matrix of T with respect to B and C, i.e.

$$\boxed{A_C^B[u]_B = [T(u)]_C} \qquad \text{for all } u \text{ in } \mathbb{R}^n.$$

Then

$$\boxed{A_C^B = P_C^{-1}\mathrm{A}P_B}.$$

Proof: <u>**Idea:**</u> Read the diagram above.

SIMILARITY = SAME TRANSFORMATION WITH DIFFERENT BASIS

Note: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with **standard matrix** A. Then A and K are similar matrices, if and only if K is the matrix of T with respect to another basis B.

Proof:

Example: Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, $B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix A.

1.) Show that the matrix $A_B^B = P_B^{-1} A P_B$ of T with respect to B (and B) is a diagonal matrix D. This means that A and D are similar.

2.) Use 1.) to find the determinant of A.

3.) Calculate A^5 in a simple way using 1.)

DIAGONALIZATION

Definition: A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if

$$\boxed{A = PDP^{-1}}$$

for some **invertible matrix** $\boxed{\boldsymbol{P} = \boldsymbol{P_B} = [\mathrm{b}_1, \ldots, \mathrm{b}_n]}$ and some **diagonal matrix** $\boldsymbol{D}$.

Theorem 5: An $n \times n$ matrix A is **diagonalizable** if and only if A has n linearly independent eigenvectors that form a basis of $\mathbb{R}^n$. We call such a basis B= $\{\mathrm{b}_1, \ldots, \mathrm{b}_n\}$ an **eigenvector basis** of $\mathbb{R}^n$.

Note 1: Theorem 5 says that if there is an eigenvector basis B of A, then the corresponding transformation

$$T: \mathbb{R}^n \to \mathbb{R}^n, x \mapsto T(x) = Ax$$

has a very **simple form** with respect to the basis B.

DIAGONALIZATION

General procedure for a diagonalization of an $n \times n$ matrix A:

1.) Find the **eigenvalues** of A by solving the equation

$$\boxed{\det(A - \lambda I_n) = 0} \quad \text{for } \lambda.$$

2.) For each **eigenvalue λ_i** find a **basis of eigenvectors B(λ_i)** for

$$\boxed{\mathrm{Eig}(A,\lambda_i) = \mathrm{Nul}(A - \lambda_i I_n)}.$$

3.) If the combined bases $(\mathrm{B}(\lambda_i))_i$ form a basis B of $\mathbb{R}^n$, then

$$\boxed{P_B = [\mathrm{B}(\lambda_1), \mathrm{B}(\lambda_2), \ldots]} \quad \text{and} \quad \boxed{A = P_B D P_B^{-1}}\ .$$

Note: Theorem 7 will give us conditions to see when a diagonalization is not possible. These conditions allow us to stop after Step 1.) or 2.)

Example: Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Note: $\det(A - \lambda I_n) = -(\lambda - 1)(\lambda + 2)^2$ and

$$\text{Nul}(A - 1I_n) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

DIAGONALIZATION

THEOREMS ABOUT DIAGONALIZATION

Theorem 6: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

- **Proof:** Let $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A. Then $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent, by **Theorem 2** in **Sect. 5.1**. Hence A is diagonalizable, by **Theorem 5**.

Note: 1.) It is **not necessary** for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable (see the previous **Example**)

2.) When A is diagonalizable but has fewer than n distinct eigen-values, it might still be possible to find a basis of eigenvectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$, to build $P_B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$.

THEOREMS ABOUT DIAGONALIZATION

Theorem 7: Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, where p $\leq n$. Let $m(\lambda_k)$ be the multiplicity of the eigenvalue λ_k.

a. For $1 \leq k \leq p$ we have: $\boxed{\dim(\mathrm{Eig}(A, \lambda_k)) \leq m(\lambda_k)}$.

b. The matrix A is diagonalizable if and only if

$$\boxed{\dim(\mathrm{Eig}(A, \lambda_1)) + \ldots + \dim(\mathrm{Eig}(A, \lambda_p)) = n} \ .$$

This happens if and only if

(i) the **char. polynomial factors completely** into linear factors

(ii) For $1 \leq k \leq p$ we have $\boxed{\mathbf{dim(Eig(A}, \boldsymbol{\lambda_k})) = \boldsymbol{m}(\boldsymbol{\lambda_k})}$.

c. If A is diagonalizable and B_k is a basis for $\mathrm{Eig}(A, \lambda_k)$, then B=$\{\mathsf{B}_1, \ldots, \mathsf{B}_p\}$ forms an eigenvector basis for $\mathbb{R}^n$.

Example 1: We have seen that the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$ is diagonalizable.

Example 2: The matrix

$$B = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$ is not diagonalizable.

Note: $\det(B - \lambda I_n) = -(\lambda - 1)(\lambda + 2)^2$ and

$$\text{Nul}(B - 1I_n) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

THEOREMS ABOUT DIAGONALIZATION