Math 22 –
Linear Algebra and its applications

- Lecture 21 -

Instructor: Bjoern Muetzel

GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, Th, **Sun 2-4 pm** in **KH 229**

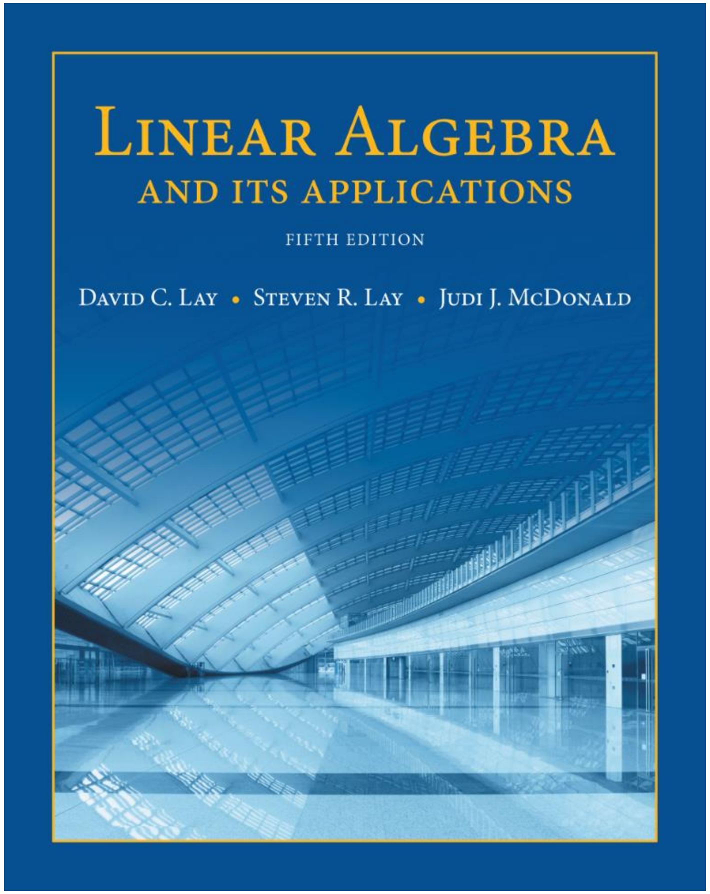
  Tutorial: Tu, Th, **Sun 7-9 pm** in **KH 105**

- **Midterm 2:** today at **4 pm** in **Carpenter 013**
  - **Topics: Chapter 2.1 – 4.7** (included)
  - about **8-9** questions
  - **Practice exam 2 solutions** available

6 Orthogonality and Least Squares

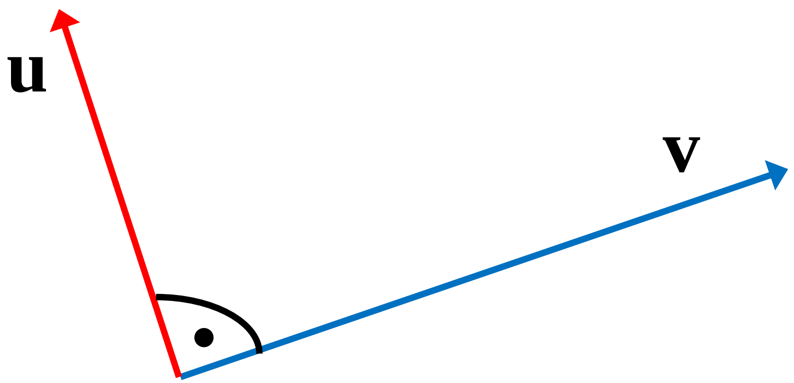
6.2

ORTHOGONAL SETS



Summary:

If the vectors of a set **S** are **orthogonal,** then they are pairwise orthogonal to each other. If these **vectors** are additionally **normalized** then they look like a standard basis.

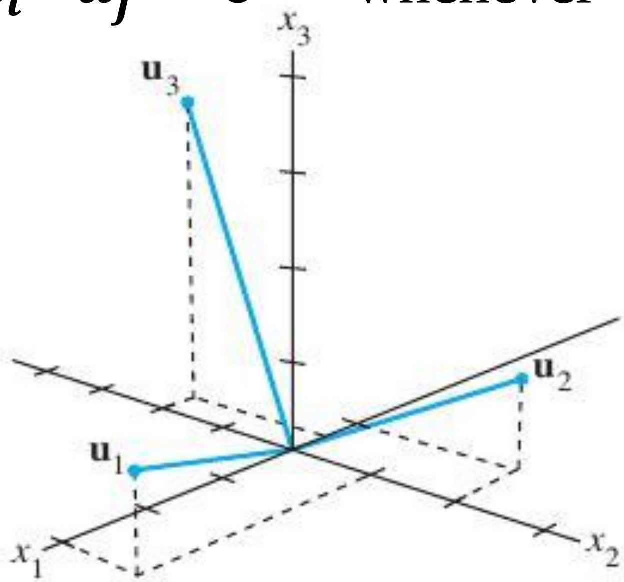


GEOMETRIC INTERPRETATION

ORTHOGONAL SETS

- **Definition:** A set of vectors $\{u_1, \ldots, u_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** if these vectors are pairwise orthogonal, that is, if

$$\boldsymbol{u_i} \cdot \boldsymbol{u_j} = \mathbf{0} \qquad \text{whenever} \qquad i \neq j.$$



- **Theorem 4:** If $S=\{u_1, \ldots, u_p\}$ is an orthogonal set of **nonzero** vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Proof:

Definition: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Note: Theorem 4 says that an orthogonal set S of nonzero vectors is automatically a basis for Span{S}.

ORTHOGONAL SETS

- **Theorem 5:** Let $\{u_1, \ldots, u_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each y in W, the weights in the linear combination are given by

$$y = x_1u_1 + \cdots + x_pu_p, \quad \text{where} \quad x_j = \frac{y \cdot u_j}{u_j \cdot u_j} \text{ for all } j \text{ in } \{1, \ldots, p\}.$$

Note: For the matrix $U = [u_1, u_2, \ldots, u_p]$ and y in W this means that we can immediately write down the **solution** $\boldsymbol{x}$ for $\mathbf{U}\boldsymbol{x} = \boldsymbol{y}$.

Proof:

Example: Let $u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$ and $U = [u_1, u_2]$.

1.) Show that u_1 and u_2 are orthogonal.

2.) For $y = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find x, such that $Ux = y$ using **Theorem 5.**

ORTHOGONAL PROJECTION

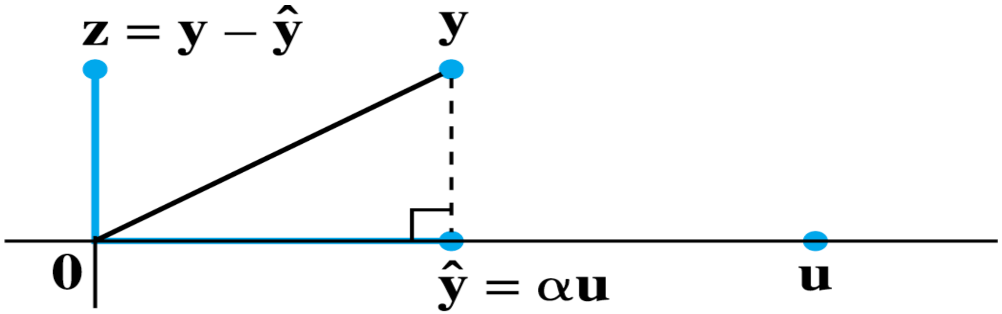
- **Theorem:** Let u be a nonzero vector in $\mathbb{R}^n$ and L= Span$\{u\}$. Then the **orthogonal projection of** a vector y in $\mathbb{R}^n$ **onto *u*** (**or *L***) is

$$\hat{y} = proj_L(y) = \frac{y \cdot u}{u \cdot u} u.$$

The **component of *y* orthogonal to *u*** is

$$z = y - \hat{y}.$$

- **Example:** Orthogonal projection in $\mathbb{R}^2$.



Finding α to make $\mathbf{y} - \hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.

- **Note:** The vectors z and $\hat{y}$ are orthogonal as $\hat{y}$ is in Span$\{u\}$ and z is orthogonal to u.

- **Proof of the Theorem:**

ORTHOGONAL PROJECTION

Example: Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$.

1.) Find the orthogonal projection $\hat{y}$ of y onto u.

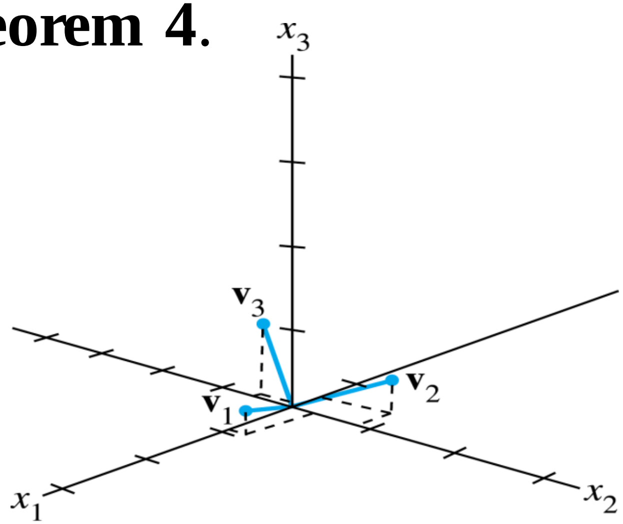
2.) Write y as the sum of the two orthogonal vectors $\hat{y}$ in Span$\{u\}$ and z, which is orthogonal to u.

3.) Draw a picture of the vectors y, u, $\hat{y}$ and z in $\mathbb{R}^2$.

ORTHONORMAL SETS

- **Definition:** A set $\{\mathbf{u}_1,\ldots,\mathbf{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors.

- If W is the subspace spanned by such a set, then $\{\mathbf{u}_1,\ldots,\mathbf{u}_p\}$ is an **orthonormal basis** for W, since the set is automatically linearly independent, by **Theorem 4**.



- **Note 1:** The simplest example of an orthonormal set is the standard basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ for $\mathbb{R}^n$ or subsets of $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$.

- **Note 2:** When the vectors in an **orthogonal set** of nonzero vectors are **normalized** to have unit length, the **new vectors will be** an **orthonormal** set.

- **Example:** Let $v_1 = \frac{1}{\sqrt{11}} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $v_3 = \frac{1}{\sqrt{66}} \cdot \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$.
  Show that $\{v_1, v_2, v_3\}$ is an orthonormal basis of $\mathbb{R}^3$, then draw a picture.

ORTHONORMAL SETS

ORTHONORMAL SETS

- **Theorem 6:** An $m \times n$ matrix U, where $n \leq m$ has orthonormal columns if and only if $U^T U = I_n$.

- **Proof:** Let $U = [u_1, u_2, \ldots, u_n]$ and compute $U^T U$.

ORTHONORMAL SETS

- **Theorem 7:** Let U be an $m \times n$ matrix, where $n \leq m$ with orthonormal columns, and let x and y be in $\mathbb{R}^n$. Then
  a. $(Ux) \cdot (Uy) = x \cdot y$.
  b. $\|Ux\| = \|x\|$.
  c. $(Ux) \cdot (Uy) = 0$ if and only if $x \cdot y = 0$.

- **Note:** Properties (a) and (c) say that the linear mapping $x \mapsto Ux$ preserves **lengths**, **distance** and **orthogonality**.

- **Proof of Theorem 7:**