Math 22 –
Linear Algebra and its applications

- Lecture 20 -

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GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, **Th**, Sun 2-4 pm in **KH 229**

  Tutorial: Tu, **Th**, Sun **7-9 pm** in **KH 105**

  Come **tomorrow** to practice for the exam.

- **Homework 6:** due **today** at **4 pm** outside **KH 008**. Please divide into the parts **A, C** and **D. Exercise 1 b)** is **optional.**

- **Midterm 2:** Friday **Nov 1** at **4 pm** in **Carpenter 013**

  - **Topics: Chapter 2.1 – 4.7** (included)

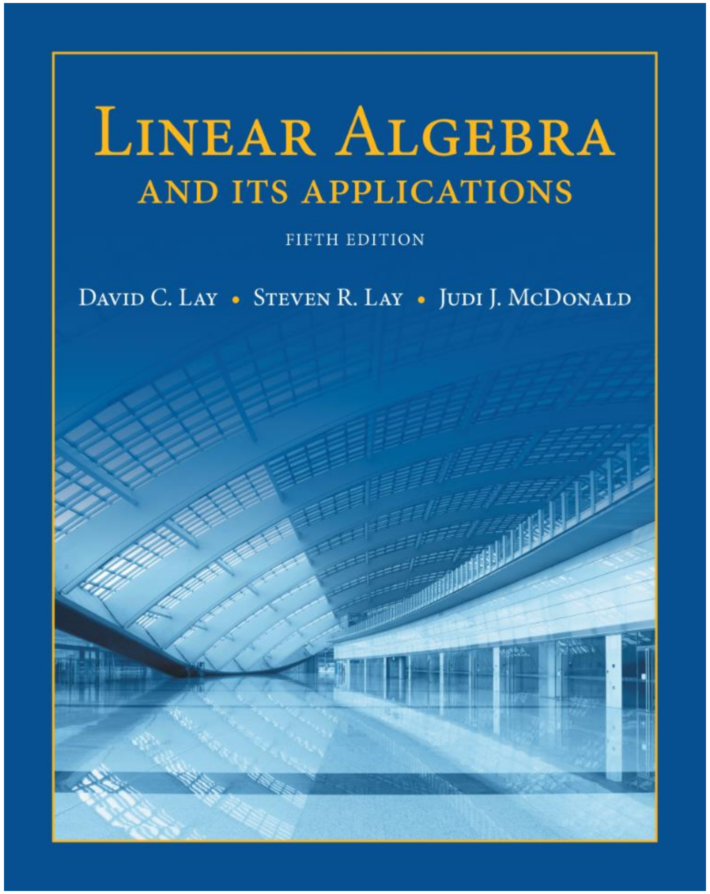
  - about **8-9** questions

  - **Practice exam 2 solutions** available

6 Orthogonality and Least Squares

6.1

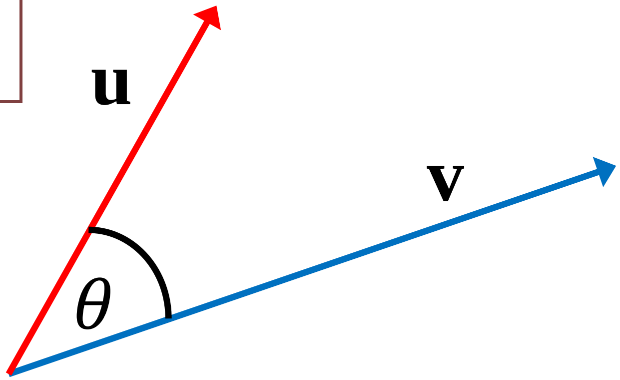
INNER PRODUCT, LENGTH, AND ORTHOGONALITY



Summary:

If **u** and **v** are vectors in $\mathbb{R}^n$ then the **inner product** $\mathbf{u}^T\mathbf{v} = \mathbf{u}\cdot\mathbf{v}$ allows us to measure **angles**, **lengths** and **distances.**

$$\cos\theta = \frac{u\bullet v}{\|u\|\|v\|}; \qquad \theta = \cos^{-1}\left(\frac{u\bullet v}{\|u\|\|v\|}\right)$$



GEOMETRIC INTERPRETATION

Inner product in $\mathbb{R}^2$

Given two vectors $\mathrm{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ how can we measure the length of these vectors and the angle between them?

INNER PRODUCT

- If **u** and **v** are vectors in $\mathbb{R}^n$, then we can regard **u** and **v** as $n \times 1$ matrices. The transpose $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which we write as a real number without brackets.

- **Definition:** If $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ are vectors in $\mathbb{R}^n$, then

  the **inner product** or **dot product** of **u** and **v** is

$$\boxed{u^T v = u \cdot v} = [u_1, u_2, \dots, u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \boxed{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}.$$

- **Theorem 1:** Let **u**, **v**, and **w** be vectors in $\mathbb{R}^n$, and let c in $\mathbb{R}$ be a scalar. Then
  a. $u \cdot v = v \cdot u$
  b. $(u + v) \cdot w = u \cdot w + v \cdot w$
  c. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
  d. $u \cdot u \geq 0$, and $u \cdot u = 0$ if and only if $u = 0$.

- **Consequence: (Linearity)**

$$\boxed{(c_1 u_1 + \ldots + c_p u_p) \cdot w = c_1(u_1 \cdot w) + \cdots + c_p(u_p \cdot w)}.$$

Proof: 1.) a. and d. can be easily checked.

2.) b. and c. are true as the dot product is a matrix multiplication which is linear. By a. it is linear from “both sides”.

3.) The consequence follows from b. and c.

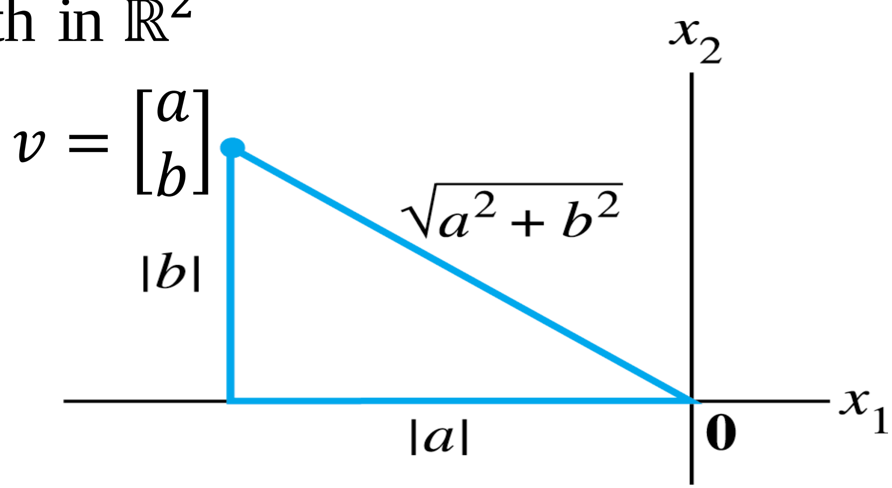
LENGTH

- If **v** is in $\mathbb{R}^n$, with entries $v_1, \ldots, v_n$, then the square root of $v \cdot v$ is well-defined as $v \cdot v \geq 0$. We define:

- **Definition:** The **length** (or **norm**) $\|v\|$ of **v** is the nonnegative real number defined by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$, hence $$\|v\|^2 = v \cdot v.$$

Example: Length in $\mathbb{R}^2$



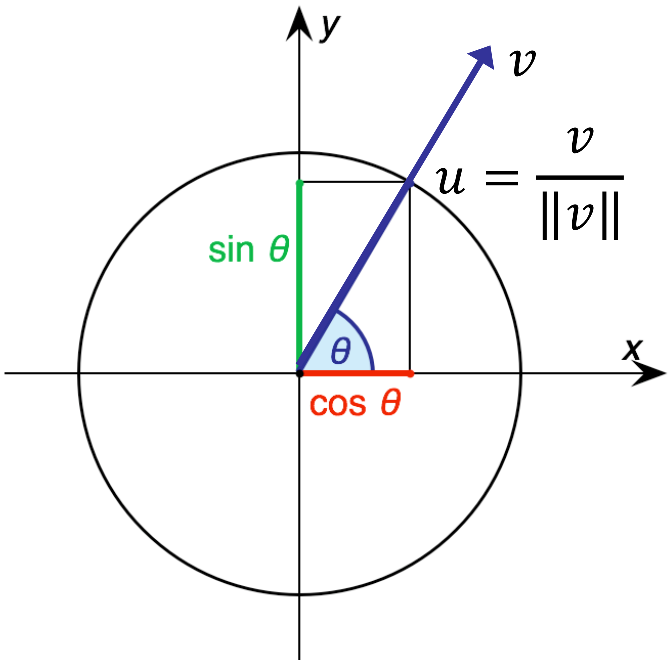
Interpretation of $\|\mathbf{v}\|$ as length.

Definition: A vector v in $\mathbb{R}^n$, that has length

$$\|v\| = \sqrt{v \cdot v} = 1 \qquad \text{is called a } \textbf{unit vector}.$$

Note: If a vector v in $\mathbb{R}^n$ is multiplied by a scalar c in $\mathbb{R}$, then

$$\|cv\| = \sqrt{cv \cdot cv} = \sqrt{c^2(v \cdot v)} = |c| \cdot \|v\|.$$



Definition: If we divide a vector $v \neq 0$ by its length $\|v\|$ then we obtain a **unit vector** $\boldsymbol{u} = \dfrac{\mathbf{1}}{\|\boldsymbol{v}\|} \cdot \boldsymbol{v}$. The process of creating u from v is called **normalizing** v, and we say that $\boldsymbol{u}$ **is in the same direction as** $\boldsymbol{v}$**.**

LENGTH

- **Example:** Let $v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ be a vector in $\mathbb{R}^3$.

  1.) Calculate $||v||$.

  2.) Find a unit vector u in the same direction as v. What is $||u||^2$?

  3.) Draw a coordinate system with v, u and the unit sphere

  $$S = \{x \text{ in } \mathbb{R}^3, ||x|| = 1\} \text{ in } \mathbb{R}^3.$$

ANGLE

Definition: Let u and v be vectors in $\mathbb{R}^n$. Then the **angle** $\boldsymbol{\theta = \measuredangle(u, v)}$ between u and v is given by

$$\boxed{\cos(\theta) = \frac{u \cdot v}{\|u\| \, \|v\|}} \quad \text{or} \quad \boxed{\theta = arccos\left(\frac{u \cdot v}{\|u\| \, \|v\|}\right)}.$$

Example: For $u = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, compute $\cos(\measuredangle(u, v))$. Estimate the angle $\measuredangle(u, v)$ with a calculator and with a sketch of u and v in $\mathbb{R}^2$.

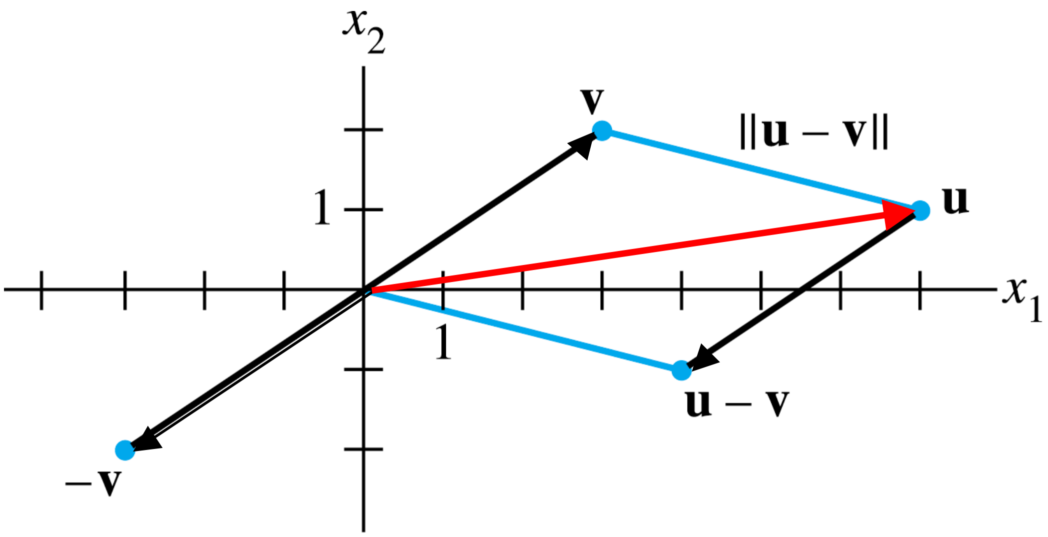
Does the angle change if we normalize u and v ?

DISTANCE

Definition: For u and v in $\mathbb{R}^n$, the **distance between $\boldsymbol{u}$ and $\boldsymbol{v}$**, written as dist(u,v), is the length of the vector u-v. That is,

$$\text{dist}(u, v) = \|u - v\|$$

Example: Distance in $\mathbb{R}^2$



The distance between **u** and **v** is the length of **u** – **v**.

Example: Compute the distance between $u = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Check your calculation with a sketch of u and v in $\mathbb{R}^2$.

ORTHOGONALITY

- **Note:** Two nonzero vectors u and v in $\mathbb{R}^n$ are **perpendicular,** i.e. $\boxed{\measuredangle(u,v) = \frac{\pi}{2}}$ or $\cos(\measuredangle(u,v)) = 0$ if and only if $\boxed{u \cdot v = 0.}$

- **Definition:** Two vectors u and v in $\mathbb{R}^n$ are **orthogonal** to each other if $\boxed{\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{0}}$.

Example: The zero vector **0** is orthogonal to every vector in $\mathbb{R}^n$.

- **Theorem 2:** Two vectors u and v in $\mathbb{R}^n$ are orthogonal if and only if

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof:

Definition: (Orthogonal Complements)

1.) If a vector **z** is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then **z** is said to be **orthogonal to** W.

2.) The set of all vectors **z** that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^{\perp}$. $W^{\perp}$ is read as "W perpendicular" or simply "W perp".

Note: 1.) A vector **x** is in $W^{\perp}$ if and only if **x** is orthogonal to every vector in a set that spans W.

2.) $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Proof: Idea: Use linearity of the inner product for 1.). Then check the subspace criteria for 2.) See **HW 7**.

ORTHOGONALITY

Theorem 3: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$\boxed{(Row\ A)^{\perp} = Nul\ A} \quad \text{and} \quad \boxed{(Col\ A)^{\perp} = Nul\ A^T}\ .$$

Proof: See **HW 7**

How can we find the orthogonal complement $W^{\perp}$ of a subspace W?

Note: Given a subspace W= Span$\{a_1, a_2, \ldots, a_m\}$ in $\mathbb{R}^n$.

Let A be the $n \times m$ matrix $A = [a_1, a_2, \ldots, a_m]$.

Theorem 3 states that $W^{\perp} = (Col\ A)^{\perp} = Nul\ A^T$.

Example: Let $\mathrm{W} = \mathrm{Span}\left\{\begin{bmatrix}3\\4\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix}\right\}$.

Find $W^{\perp}$, then sketch W and $W^{\perp}$ in $\mathbb{R}^3$. **Hint:** Use **Theorem 3**.