Math 22 –
Linear Algebra and its applications

- Lecture 19 -

Instructor: Bjoern Muetzel

GENERAL INFORMATION

- **Office hours:** **Tu 1-3 pm**, Th, Sun 2-4 pm in **KH 229**

  Tutorial: **Tu**, Th, Sun **7-9 pm** in **KH 105**

- **Homework 6:** due **Wednesday** at **4 pm** outside **KH 008**. Please divide into the parts **A, C** and **D. Exercise 1 b)** is **optional.**

- **Wednesday:** Quiz!

- **Midterm 2:** Friday **Nov 1** at **4 pm** in **Carpenter 013**

  - **Topics: Chapter 2.1 – 4.7** (included)

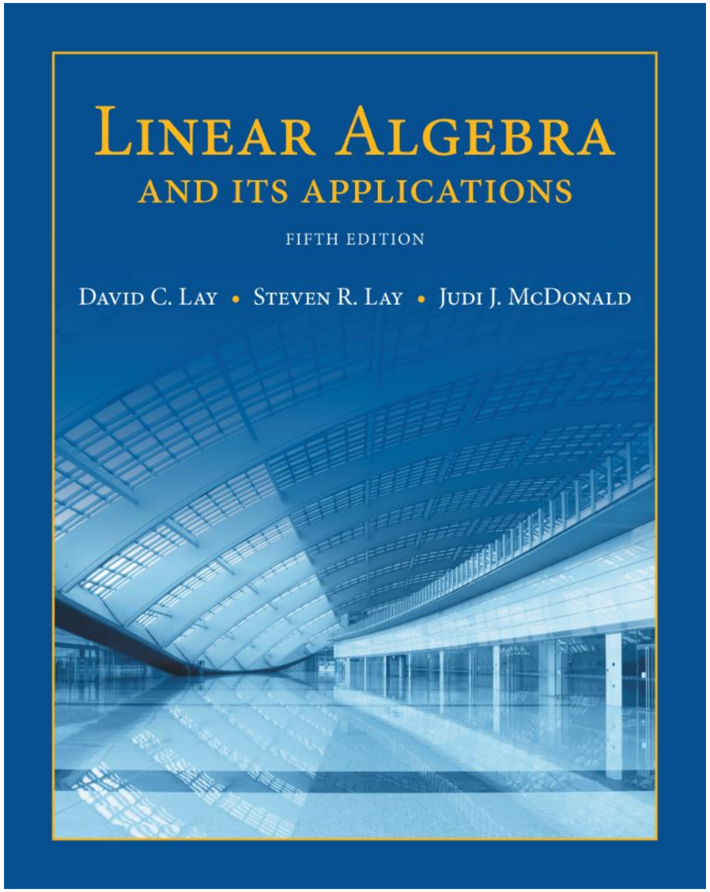
  - about **8-9** questions

  - **Practice exam 2** available on Sunday

4 Vector Spaces

4.7

CHANGE OF BASIS



Summary:

Given a description of a **vector** with respect to **two different bases** then the **change-of-coordinates matrix** allows us to switch from one description to another.

GEOMETRIC INTERPRETATION

Example: Consider the two bases for $\mathbb{R}^2$

$$B = \{b_1, b_2\} = \left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\} \text{ and } C = \{c_1, c_2\} = \left\{\begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}, \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}\right\}$$

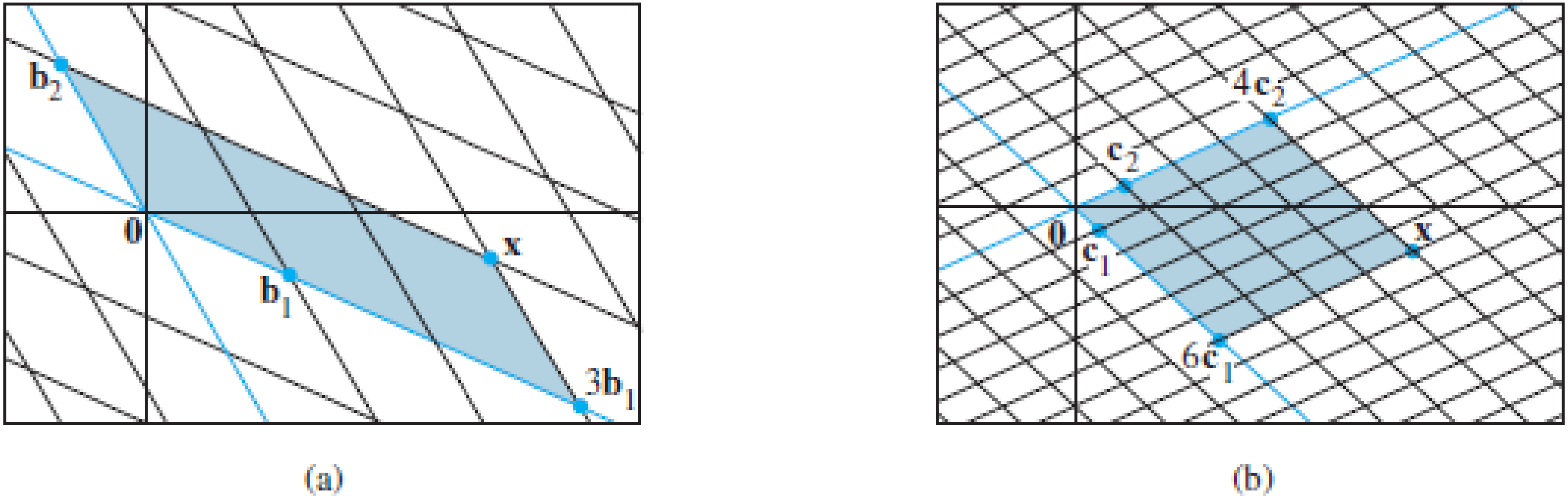


FIGURE 1 Two coordinate systems for the same vector space.

Question: Let **x** in $\mathbb{R}^2$ be given in B-coordinates $[x]_B$. What are $[x]_C$?

GEOMETRIC INTERPRETATION

CHANGE OF BASIS IN $\mathbb{R}^n$

Let $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$ in $\mathbb{R}^n$ be two different bases.

How can we pass from B-coordinates to C-coordinates and vice versa?

- If $B = \{b_1, \ldots, b_n\}$ and $E = \{e_1, \ldots, e_n\}$ is the standard basis in $\mathbb{R}^n$, then we know the answer. We have seen in **Lecture 16**:

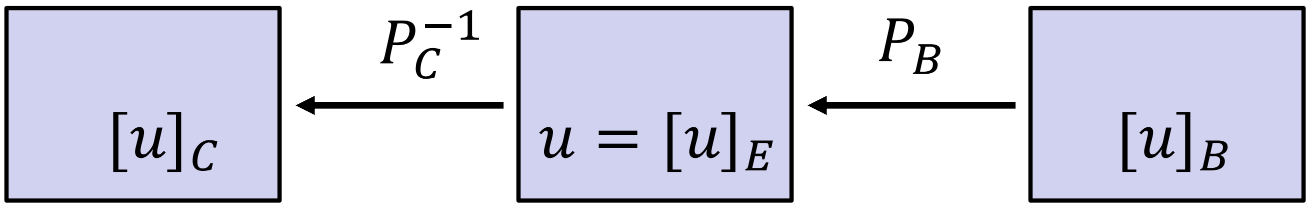
- **Reminder:** $P_B = [b_1, \ldots, b_n]$ is the **change-of-coordinates matrix** from B to the standard basis E in $\mathbb{R}^n$. For any u in $\mathbb{R}^n$

$$\boxed{u = [u]_E = P_B[u]_B} \quad \text{and} \quad \boxed{[u]_B = P_B^{-1}[u]_E = P_B^{-1}u}$$

and therefore P_B^{-1} is a **change-of-coordinate matrix** from E to B.

This means we can pass from B-coordinates to E-coordinates and then to C-coordinates:

$$u = P_B[u]_B \quad \text{and} \quad [u]_C = P_C^{-1}u \quad \text{hence} \quad [u]_C = P_C^{-1} \cdot P_B[u]_B.$$



Theorem: If $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$ in $\mathbb{R}^n$ are two different bases, then the coordinates $[u]_C$ and $[u]_B$ satisfy:

$$\boxed{[u]_C = P_C^{-1} \cdot P_B[u]_B = P_C^B[u]_B}$$

We call $\boxed{P_C^B = P_{C\leftarrow B}}$ the **change-of-coordinates matrix from B to *C***.

CHANGE OF BASIS IN $\mathbb{R}^n$

- **Example:** Consider the two bases in $\mathbb{R}^2$

$$B = \{b_1, b_2\} = \left\{\begin{bmatrix} -9 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \end{bmatrix}\right\} \quad \text{and } C = \{c_1, c_2\} = \left\{\begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix}\right\}.$$

Find the change-of-coordinates matrix $P_C^B = P_{C \leftarrow B}$ from B to *C*.

CHANGE OF BASIS IN GENERAL

- **Theorem 15:** Let $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$ be bases for a vector space $\underline{V}$. Then there is a unique n × n matrix $P_C^B = P_{C \leftarrow B}$ such that

$$\boxed{[x]_C = P_C^B [x]_B} \qquad (1)$$

- The columns of P_C^B are the C-coordinate vectors of the vectors in the basis B. That is

$$\boxed{P_C^B = [\ [b_1]_C, [b_2]_C, \ldots, [b_n]_C]} \qquad (2)$$

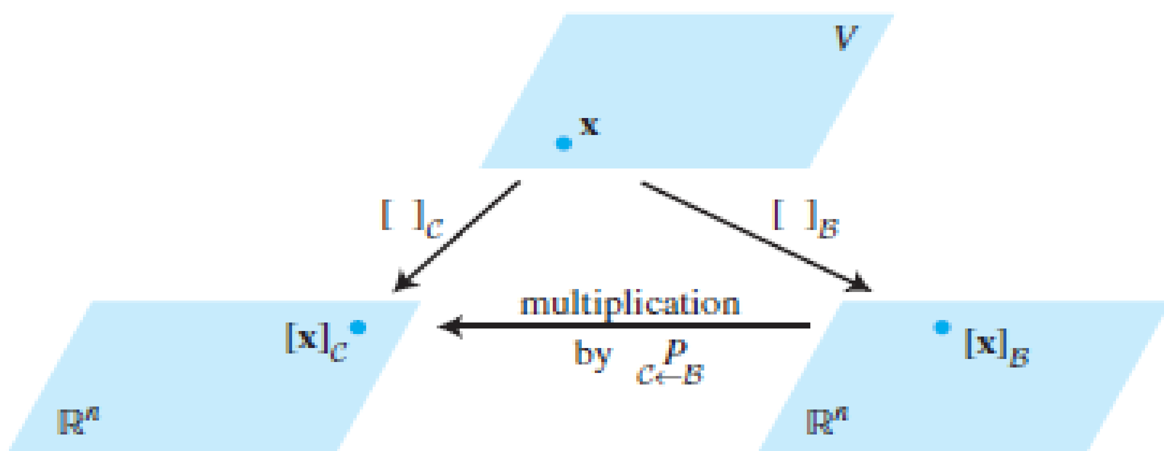


FIGURE 2 Two coordinate systems for V.

Proof: Follows from the linearity of the coordinate map.

Definition: The matrix $P_C^B = P_{C \leftarrow B}$ in **Theorem 15** is called the **change-of-coordinates matrix from B to *C***.

Multiplication by P_C^B converts *B*-coordinates into *C*-coordinates:

Theorem: Let $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$ be bases for a vector space *V*. Then

a) $\boxed{P_B^C = P_{B \leftarrow C} = (P_{C \leftarrow B})^{-1} = (P_C^B)^{-1}}$ or

b) $[x]_C = P_C^B [x]_B$ and $[x]_B = (P_C^B)^{-1} [x]_C$

<u>**Note:**</u> To change coordinates between two **nonstandard bases** in $\mathbb{R}^n$, we need **Theorem 15**. The theorem shows that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

CHANGE OF BASIS IN GENERAL

Proof of the Theorem:

1.) The columns of P_B^C are linearly independent because they are the coordinate vectors of the linearly independent set *B*. We have by (1)

$$[x]_C = P_C^B\,[x]_\mathrm{B}.$$

2.) Since P_C^B is square, it must be invertible, by the Invertible Matrix Theorem. Left-multiplying both sides of Equation (1) by its inverse matrix $(P_C^B)^{-1}$ yields

$$(P_C^B)^{-1}[x]_C = [x]_\mathrm{B}.$$

3.) Thus $(P_C^B)^{-1}$is the matrix that converts C-coordinates into B–coordinates. That is $(P_C^B)^{-1} = P_B^C$.

Example: Consider the bases $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ of a two-dimensional subspace V in $\mathbb{R}^{100}$. Let u in V be given in B-coordinates by $[u]_B = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$. We know that $b_1 = c_1 - c_2$ and $b_2 = c_1 + c_2$.

1.) Find the C-coordinates of u.

2.) Write down $P_C^B = P_{C \leftarrow B}$ and calculate $P_B^C = P_{B \leftarrow C}$.