Math 22 –
Linear Algebra and its applications

- Lecture 18 -

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GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, Th, **Sun 2-4 pm** in **KH 229**
  Tutorial: Tu, Th, **Sun 7-9 pm** in **KH 105**

- **Homework 6:** due **next Wednesday** at **4 pm** outside **KH 008**. Please divide into the parts **A, C** and **D. Exercise 1 b)** is **optional.**

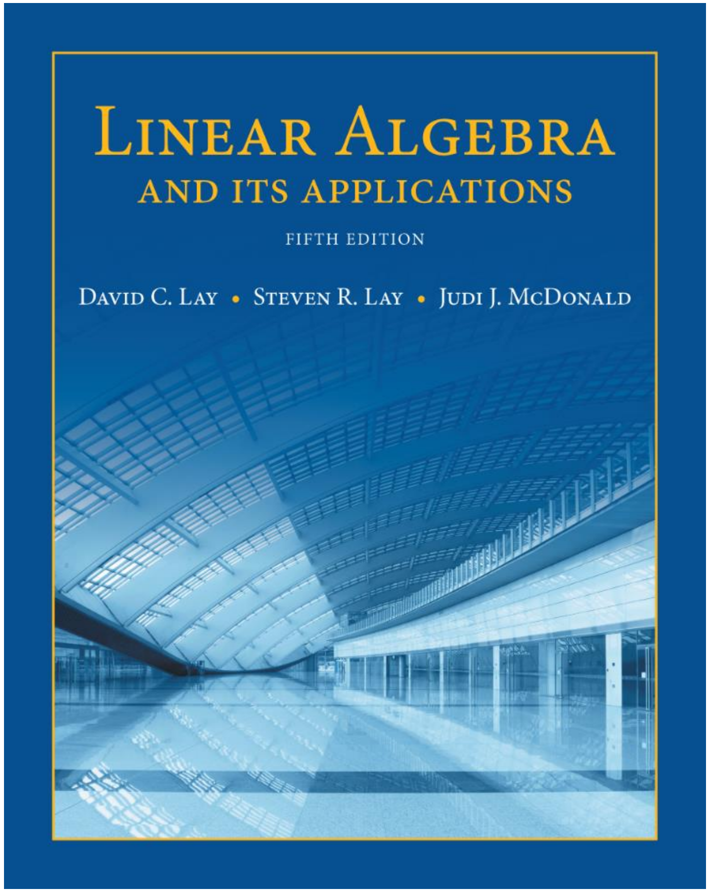
- **Project:** Meeting this **weekend**!

- **Midterm 2:** Friday **Nov 1** at **4 pm** in **Carpenter 013**
  - **Topics: Chapter 2.1 – 4.7** (included)
  - about **8-9** questions
  - **Practice exam 2** available on Sunday

4 Vector Spaces

4.6

RANK OF A MATRIX



- **Summary:**

1.) We can consider the rows of a matrix as **row vectors**. The **row space** is the vector space spanned by these vectors.

2.) The **Rank Theorem** sums up the relation between the **null space**, the **column space** and the **rank** of a matrix.

THE ROW SPACE

- If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$.

- **Definition:** The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by Row A. Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$. Clearly

$$\boxed{\text{Row } A = \text{Col } A^{\mathrm{T}}}$$

Motivation: We will see later that $\boxed{\mathbb{R}^n = \text{Nul } A + \text{Row } A.}$

Note: As Row $A = \text{Col } A^{\mathrm{T}}$ clearly, the pivot columns of A^{T} form a basis of Row A.

Figure:

However, when we have already calculated the echelon form U of A, then there is an <u>easier way</u> to get a **basis for Row *A***:

Theorem 13:

a) If two matrices A and B are row equivalent, then their row spaces are the same.

b) If U is the echelon form of A, then the nonzero rows of U form a basis for the row space of A as well as for that of U. Furthermore

$$\textbf{dim Row } A = \textbf{dim Col } A$$

Warning: This time the **rows of** the **echelon form** U of A **span the row space Row *A***, not the rows of A itself.

THE ROW SPACE

Proof: 1.)We noticed in the last lecture that elementary column operations do not change the column space of a matrix.

2.) As Col A^T = Row A this means that elementary row operations do not change the row space Row A.

3.) The non-zero rows of U are linearly independent (as row vectors)

4.) Then number of non-zero rows of U is equal to the number of pivot columns of U, hence dim Row A = dim Col A.

THE ROW SPACE

- **Example:** Find bases for the row space, the column space, and the null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

and determine the dimensions of Row A, Col A and Nul A.

THE ROW SPACE

Observation: Unlike the basis for Col A, the bases for Row A and Nul A have no simple connection with the entries in A itself.

THE RANK THEOREM

- **Definition:** The **rank** of A is the dimension of the column space of A.

$$\boxed{\text{rank } A = \dim \text{Col } A}$$

Since Row A is the same as Col A^T, we have

$$\boxed{\text{rank } A^T = \dim \text{Col } A^T = \dim \text{Row } A}$$

- **Theorem 14*:** Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map and A be the corresponding standard matrix. Then

a) rank A^T = dim Row A = dim Col A = rank A , hence

b) rank A + dim Nul A = n or equally

c) $\boxed{\dim\ T(\mathbb{R}^n) + \dim \text{Nul}(T) = \dim\,(\mathbb{R}^n)}$

Proof: The dimensions of these spaces can be deduced from the number of pivot or non-pivot columns of A (see **Theorem 6** and **13**).

THE RANK THEOREM

Consequence: As the number of pivots can not exceed the number of rows or columns, **Theorem 14*** implies the following two inequalities:

1.) dim Col A = rank A $\leq$ min{m,n}

2.) n-m $\leq$ dim Nul A $\leq$ n – dim Col A

Note: This list could be easily further expanded. However, we will not do this here as every new inequality would be just a logical consequence of the restrictions on the number of pivot or non-pivot columns.

- **Example:**
  a. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?
  b. Could a 6×9 matrix B have a two-dimensional null space?

THE INVERTIBLE MATRIX THEOREM (CONTINUED)

- **Theorem:** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

  m. The columns of A form a basis of $\mathbb{R}^n$.

  n. Col $A = \mathbb{R}^n$

  o. dim Col A = n

  p. Rank A = n

  q. Nul $A = \{0\}$

  r. dim Nul $A = 0$

  Proof: This follows from **Theorem 14***.

  You do **not** have to memorize this theorem!