Math 22 –
Linear Algebra and its applications

- Lecture 17 -

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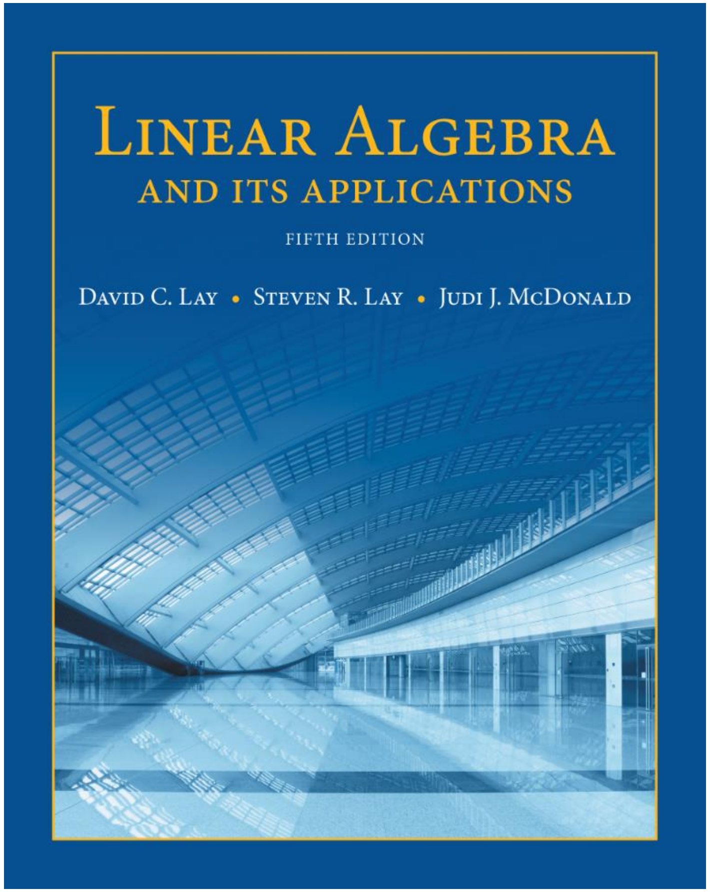
GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, **Th**, Sun **2-4 pm** in **KH 229**
- **Tutorial:** Tu, **Th**, Sun **7-9 pm** in **KH 105**
- **Homework 5:** due **today** at **4 pm** outside **KH 008**. Please divide into the parts **A, B, C** and **D** and **write your name** on each part.
- **Project:** Meeting next **weekend**!
- **Midterm 2:** Friday **Nov 1** at **4 pm** in **Carpenter 013**

4 Vector Spaces

4.5

THE DIMENSION OF A VECTOR SPACE



- **<u>Summary:</u>**

  1.) The number **n** of basis vectors of a vector space *V* is an invariant. It is called the **dimension of *V***

  2.) If the dimension **n** of a vector space is known then we can find a **basis** by finding a **spanning set** <u>or</u> finding a **linearly independent set** of n vectors.

GEOMETRIC INTERPRETATION

DIMENSION OF A VECTOR SPACE

- **Reminder:** A basis of a vector space *V* is a linearly independent set that spans *V*.

- **Theorem 9:** If a vector space *V* has a basis B = {$b_1, \ldots, b_n$}, then any set in *V* containing more than *n* vectors must be linearly dependent.

- **Theorem 10:** If a vector space *V* has a basis of *n* vectors, then every basis of *V* must consist of exactly *n* vectors.

Proof: Idea for Theorem 9: As $V \cong \mathbb{R}^n$ we can use the coordinate map

$$[\,\cdot\,]_B = T_B : V \to \mathbb{R}^n, \mathrm{x} \mapsto T_B(x) = [x]_B$$

and do all calculations in $\mathbb{R}^n$.

DIMENSION OF A VECTOR SPACE

DIMENSION OF A VECTOR SPACE

- **Definition:** If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V.

$$\mathbf{dim}\ \boldsymbol{V} = \#\ (\text{basis vectors of } V)$$

- The dimension of the zero vector space **{0}** is defined to be zero.

- If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

DIMENSION OF A VECTOR SPACE

- **Example 3:** Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

BUILDING A BASIS FROM A LINEARLY INDEPENDENT SET

- **Theorem 11:** Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set S in H can be expanded, if necessary, to a basis for H.

- **Proof:** Idea: If the set S does not span H, then we can add a vector u from H that is not in span S. $\{S \cup u\}$ is linearly independent.

- **Consequence:** Let H be a subspace of a finite-dimensional vector space V. Then

$$\boxed{\dim H \leq \dim V}$$

- **Proof:** A basis B of H is linearly independent. By **Theorem 9** the basis B can not have more vectors than dim V. Hence dim $H \leq$ dim V.

BUILDING A BASIS FROM A LINEARLY INDEPENDENT SET

How can we complete a basis B = {b_1, …, b_k} of a subspace H in $\mathbb{R}^n$ to a basis of $\mathbb{R}^n$?

Solution: Let E ={e_1, …, e_n} be the standard basis of $\mathbb{R}^n$.

1.) Write B in matrix form B=[b_1, …, b_k]. B has a pivot in every **column**.

2.) Now bring B^T into echelon form U^T. U^T has a pivot in every **row**.

3.) For any **non-pivot column j** of U^T add a row $\boldsymbol{e_j^T}$ to U^T. We call the matrix obtained this way U'^T.

4.) Then B together with the basis vectors $\mathbf{e_j}$ from 3.) forms a basis of $\mathbb{R}^n$.

BUILDING A BASIS FROM A LINEARLY INDEPENDENT SET

Proof:

BUILDING A BASIS FROM A LINEARLY INDEPENDENT SET

Example: Let $b_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Complete $\{b_1, b_2\}$ to a basis for $\mathbb{R}^3$.

THE BASIS THEOREM

- **Theorem 12: (Basis theorem)** Let *V* be an *n*-dimensional vector space, where n ≥ 1. Then

  1.) Any linearly independent set of exactly *n* elements in *V* is automatically a basis for *V*.

  2.) Any set of exactly *n* elements that spans *V* is automatically a basis for *V*.

- **Proof:** 1.) Follows from **Theorem 11** or **"pivot in every** "

  2.) Follows from the **Spanning Set Theorem** or

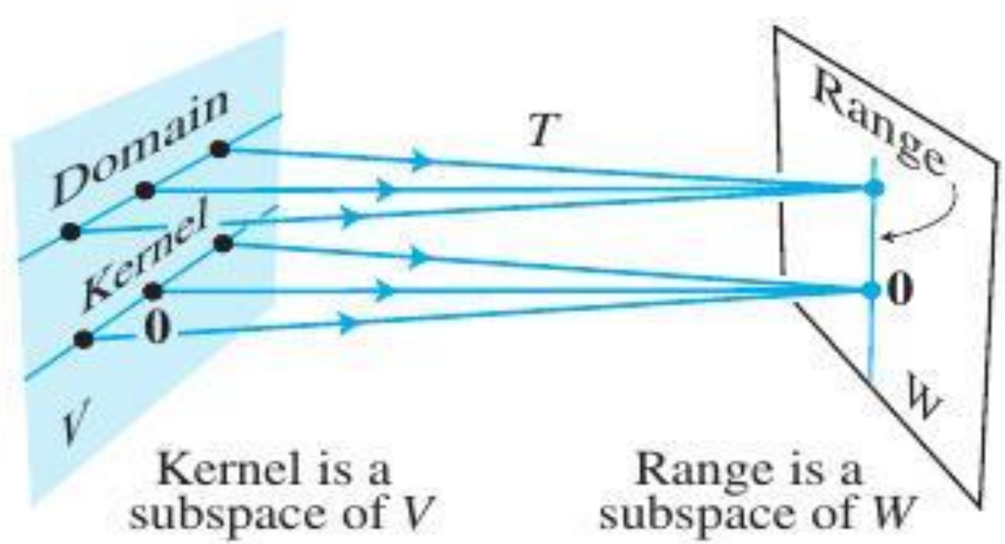
  "pivot in every "

REMINDER: KERNEL AND RANGE

- **Definition:** Let $T: V \rightarrow W, x \mapsto T(x)$ be a linear transformation.

  1.) The **kernel** (or **null space**) $Nul(T)$ of such a T is the set of all **u** in V such that $T(u) = \mathbf{0}$ in W.

  2.) The **range** $T(V)$ of T is the set of all vectors in W of the form T (**x**) for some **x** in V.



REMINDER: BASES FOR NUL *A* AND COL *A*

- **Theorem 6**: Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the corresponding standard matrix. Then the **p pivot columns of *A*** form a **basis** for

  $T(\mathbb{R}^n) =$ Col *A* ={b in $\mathbb{R}^m$, s. t. Ax = b for some x in $\mathbb{R}^n$}.

- **Warning:** The pivot columns of a matrix *A* can only be read from the echelon form U of A. But be careful to use the **pivot columns of *A*** itself for the **basis** of **Col *A***.

- **Consequence:**

  dim $T(\mathbb{R}^n)$ $=$ **dim Col *A*** = *p* = #(**pivot** columns of A)

REMINDER: BASES FOR NUL *A* AND COL *A*

We have seen in Lecture 13:

- **Theorem:** Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let *A* be the corresponding standard matrix. If

  $$\text{Nul(T)} = \text{Nul } A = \{\text{x in } \mathbb{R}^n, \text{Ax=0}\}$$

  contains nonzero vectors then a **basis for Nul *A*** consist out of **q** vectors, where **q** equals the **number of non-pivot columns of A**.

- **Reminder:** We can find Nul A explicitly by solving the homo-geneous system of linear equations Ax=0.

- **Consequence:**

  dim Nul(T)**= dim Nul *A*** = *q* = #(**non-pivot** columns of A)

THE DIMENSIONS OF NUL *A* AND COL *A*

Consequence:

$$\dim \text{Col } A + \dim \text{Nul } A = n = \dim(\mathbb{R}^n) \quad \text{or}$$

$$\boxed{\dim T(\mathbb{R}^n) + \dim \text{Nul}(T) = \dim \mathbb{R}^n}$$

DIMENSIONS OF NUL *A* AND COL *A*

Example: Let $T: \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

1.) Find a basis for $T(\mathbb{R}^5)$ = Col A and determine dim Col A.

2.) What is the dimension of Nul A=Nul(T)

3.) Find a basis for Nul A and complete this basis to a basis for $\mathbb{R}^5$