Math 22 – Linear Algebra and its applications

- Lecture 15 -

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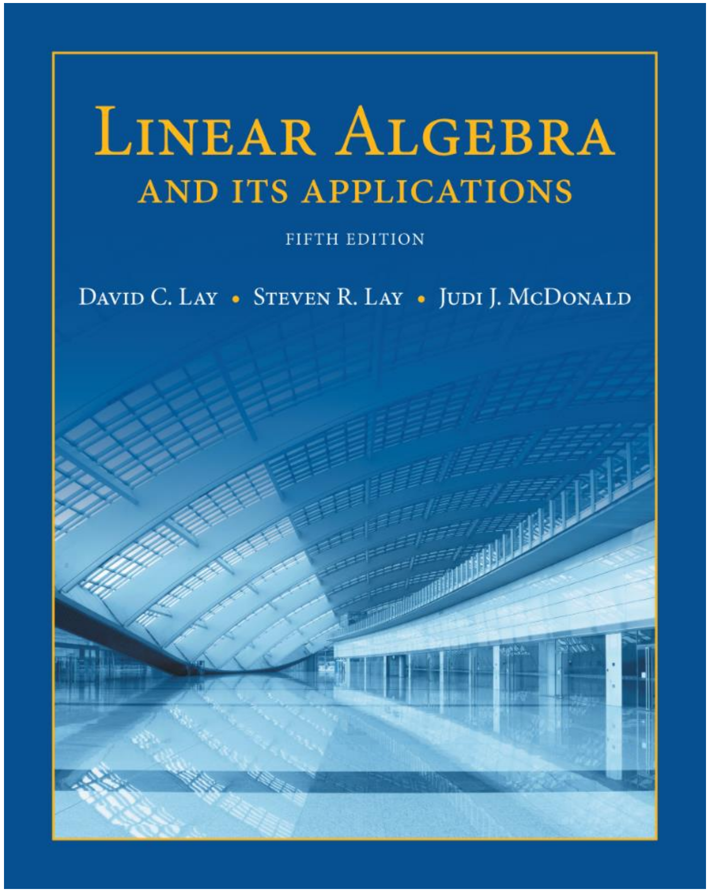
GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, Th, **Sun 2-4 pm** in **KH 229**
- **Tutorial:** Tu, **Th**, **Sun 7-9 pm** in **KH 105**
- **Homework 5:** due **next Wednesday** at **4 pm** outside **KH 008**. Please divide into the parts **A, B, C** and **D** and **write your name** on each part.

4 Vector Spaces

4.3

LINEAR INDEPENDENCE AND BASES



- **Summary:**

  1.) A **linearly independent spanning set** is called a **basis**.

  2.) We can **find** a **basis** by **eliminating** vectors from a Span or by using the **row reduction algorithm**.

GEOMETRIC INTERPRETATION

LINEAR INDEPENDENCE

- **Definition:** An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\underline{V}$ is said to be **linearly independent** if the vector equation

$$\boxed{x_1\mathrm{v}_1 + x_2\mathrm{v}_2 + \ldots + x_p\mathrm{v}_p = 0}$$

has **only** the **trivial** solution $\mathbf{x_1 = x_2 = \ldots = x_p = 0}$.

Otherwise the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent.** This means that there exist weights $c_1, \ldots, c_p$, **not all zero**, such that

$$\boxed{c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \ldots + c_p\mathrm{v}_p = 0} \quad (1)$$

Equation (1) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$ when the weights are not all zero.

- **Theorem:** The set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\underline{V}$ is **linearly independent** if and only if any vector $\mathbf{v}$ in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ has a unique description as linear combination of the $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

- **Proof:**

BASES

As this is a very practical property we define :

Definition 1: Let $S=\{v_1, \ldots, v_p\}$ be a set of vectors in *V*.
Then a subset B of vectors of S is called a **basis** of $\text{Span}\{v_1, \ldots, v_p\}$, if the vectors in B are **linearly independent** but still **span** $\text{Span}\{v_1, \ldots, v_p\}$.

Definition 2: Let *H* be a subspace of a vector space *V* and $B= \{b_1, \ldots, b_m\}$ be a subset of *H*. Then B is a **basis** of *H* if

i.) B spans *H*, .i.e. $\boxed{\text{Span}\{b_1, \ldots, b_m\}=H}$

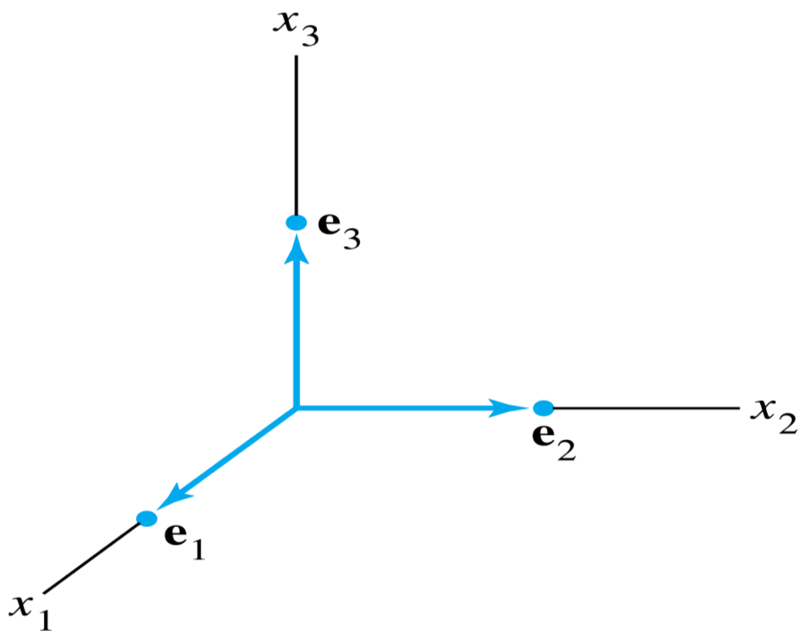
ii.) The vectors $\{b_1, \ldots, b_m\}$ are **linearly independent**.

Note: This means that **every** vector in *H* can be expressed **uniquely** as a **linear combination** of vectors in B

- **Example:** Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ matrix, I_n.
- That is,

$$\mathrm{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathrm{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathrm{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$.



The standard basis for $\mathbb{R}^3$.

BASES

Example: Find a simple basis for

1.) the set of 2x2 matrices

2.) the set of polynomials $\mathbf{P}_3 \cong \mathbb{R}^4$ of degree smaller or equal to 3.

3.) the set of polynomials $\mathbf{P}_2 \cong \mathbb{R}^3$ of degree smaller or equal to 2.

4.) Check if the polynomials B=$\{1 + x, x + x^2, x^2+1\}$ form a basis of $\mathbf{P}_2$.

HOW TO FIND A BASIS FOR A SPAN

We know:

- **Theorem: (Characterization of Linearly Dependent Sets)**
  An indexed set $S=\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors in $\underline{V}$ is **linearly dependent** if and only if at least one of the vectors in S is a linear combination of the others.

Proof: As in **Lecture 6**, **Theorem 7.**

<u>Question:</u> If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent, **can we remove a vector** $\mathbf{v}_j$ such that

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_p\} \ \mathbf{?}$$

Solution: We can remove a vector $\mathbf{v_j}$ **that has a non-zero weight** in the **linear dependence** relation.

Consequence: We can **find a basis** by successively **removing vectors** from a spanning set.

By the previous theorem the above solution is equal to

Theorem 5: (Spanning set theorem) Let $S=\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H=\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. If $\mathbf{v}_j$ is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_j$ still spans H.

Proof: as in **Lecture 6, Theorem 7 .** This proof can be directly translated into general vector spaces.

HOW TO FIND A BASIS FOR A SPAN

- **Example:** Let $v_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$

  and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that $v_3 = 5v_1 + 3v_2$.

  1.) Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

  2.) Find a basis for the subspace H.

- **Solution:**

HOW TO FIND A BASIS FOR A SPAN

In the case where $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ are vectors in $\mathbb{R}^m$, we can immediately identify a basis using the **row reduction algorithm**:

Theorem: Let S=$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^m$ and let $A = [\mathbf{v}_1, \ldots, \mathbf{v}_p]$ be the matrix, whose columns are the $\mathbf{v}_j$. Then the vectors which form the **pivot columns of A** are form a **basis** of Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Proof:

BASES FOR NUL *A* AND COL *A*

Translating this theorem into matrix notation we get:

- **Theorem 6**: Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let A be the corresponding standard matrix. Then the **p pivot columns of *A*** form a **basis** for

  $T(\mathbb{R}^n) =$ Col *A* ={b in $\mathbb{R}^m$, s. t. Ax = b for some x in $\mathbb{R}^n$}.

- **Warning:** The pivot columns of a matrix *A* can only be read from the echelon form U of A. But be careful to use the **pivot columns of *A*** itself for the **basis** of **Col *A***.

BASES FOR NUL *A* AND COL *A*

We have seen in Lecture 14:

- **Theorem:** Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let *A* be the corresponding standard matrix. If

$$\text{Nul(T)} = \text{Nul}\, A = \{\mathbf{x} \text{ in } \mathbb{R}^n, \text{A}\mathbf{x}=0\}$$

contains nonzero vectors then a **basis for Nul *A*** consist out of **q** vectors, where **q** equals the **number of non-pivot columns of A**.

- **Reminder:** We can find Nul A explicitly by solving the homogeneous system of linear equations A**x**=0.

- **Surprise:**

BASIS OF COL A AND NUL A

- **Example:**

1.) Find a basis for Span{$\mathbf{b}_1, \ldots, \mathbf{b}_5$}, where

$$B = \begin{bmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \cdots & \mathrm{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.) Find a basis for Nul B.

SUMMARY

- **Two Views of a Basis**

- When the Spanning Set Theorem is used, the deletion of vectors from a spanning set must stop when the set becomes linearly independent.

- If an additional vector is deleted, it will not be a linear combination of the remaining vectors, and hence the smaller set will no longer span V.

- **View 1:** A **basis** is a **spanning set** that is **as small as possible**.

View 2: A **basis** is also a **linearly independent set** that is **as large as possible.**

- If S is a basis for V, and if S is enlarged by one vector—say, $\mathbf{w}$—from V, then the new set cannot be linearly independent, because S spans V, and $\mathbf{w}$ is therefore a linear combination of the elements in S.