Math 22 –
Linear Algebra and its applications

- Lecture 14 -

Instructor: Bjoern Muetzel

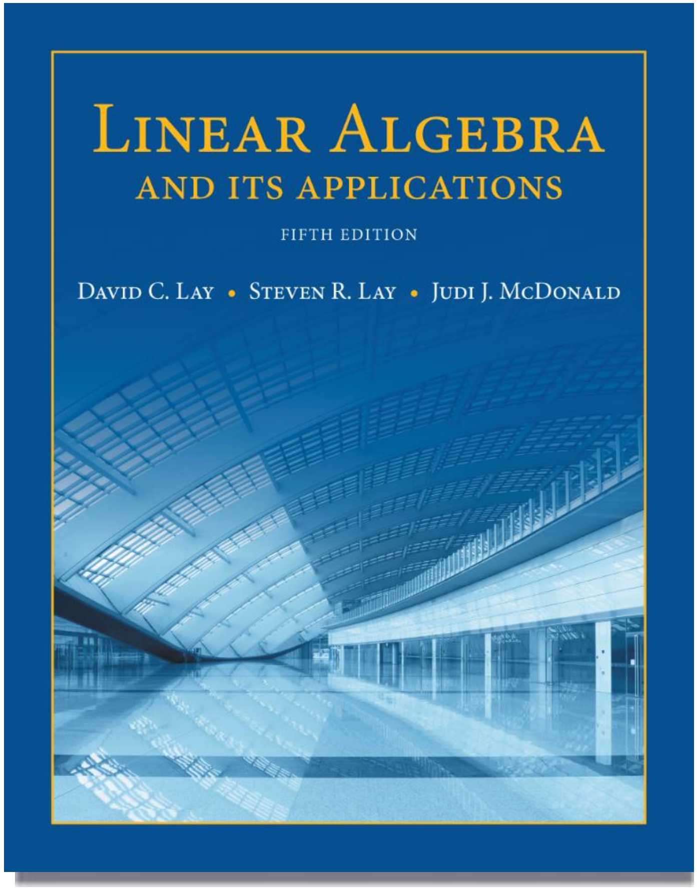
GENERAL INFORMATION

- **<u>Office hours:</u>** Tu 1-3 pm, **Th**, Sun **2-4 pm** in **KH 229**
- **<u>Tutorial:</u>** Tu, **Th**, Sun **7-9 pm** in **KH 105**
- **<u>Homework 4:</u>** due **today** at **4 pm** outside **KH 008**. Please divide into the parts **A, B, C** and **D** and **write your name** on each part.

4 Vector Spaces

4.2

NULL SPACES, COLUMN SPACES AND LINEAR TRANSFORMATIONS



Summary:

1.) The **kernel** and the **range** of a **linear transformation** are **subspaces** and carry **important information** about the map itself.

2.) In **matrix notation** the **kernel** is called the **null space** and the **range** the **column space**.

GEOMETRIC INTERPRETATION

GEOMETRIC INTERPRETATION

LINEAR TRANSFORMATION

- **Definition:** A **linear transformation** from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that for all $\mathbf{u}$, $\mathbf{v}$ in V and all scalars c in $\mathbb{R}$.

  i. $T(u+v) = T(u) + T(v)$

  ii. $T(cu) = cT(u)$

- We write shortly: $T: V \to W, x \mapsto T(x)$.

- From **i.** and the vector space axioms it **follows** that

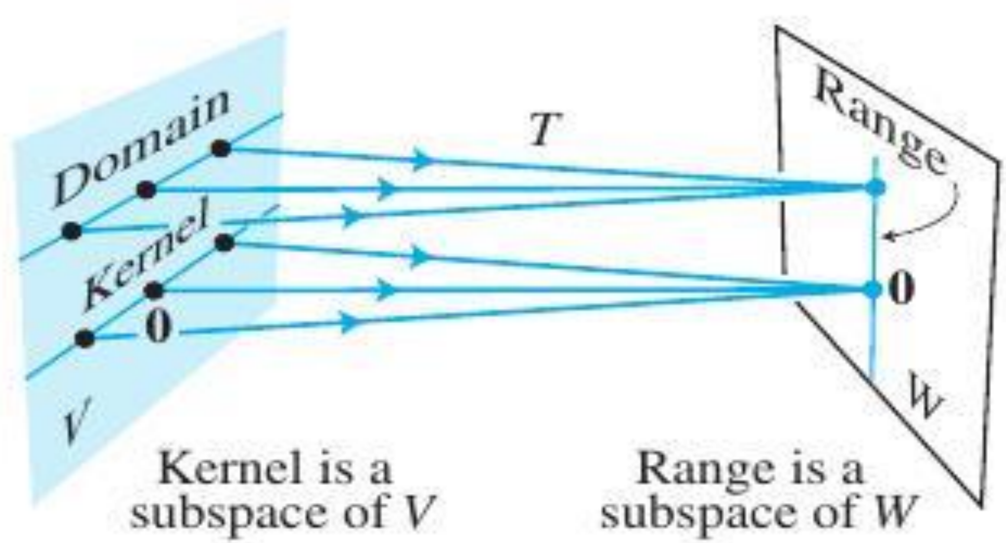
  iii. $T(\mathbf{0}) = \mathbf{0}$.

KERNEL AND RANGE

- **Definition:** Let $T: V \rightarrow W, x \mapsto T(x)$ be a linear transformation.

  1.) The **kernel** (or **null space**) $Nul(T)$ of such a T is the set of all **u** in V such that $T(u) = \mathbf{0}$ in W.

  2.) The **range** $T(V)$ of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some **x** in V.

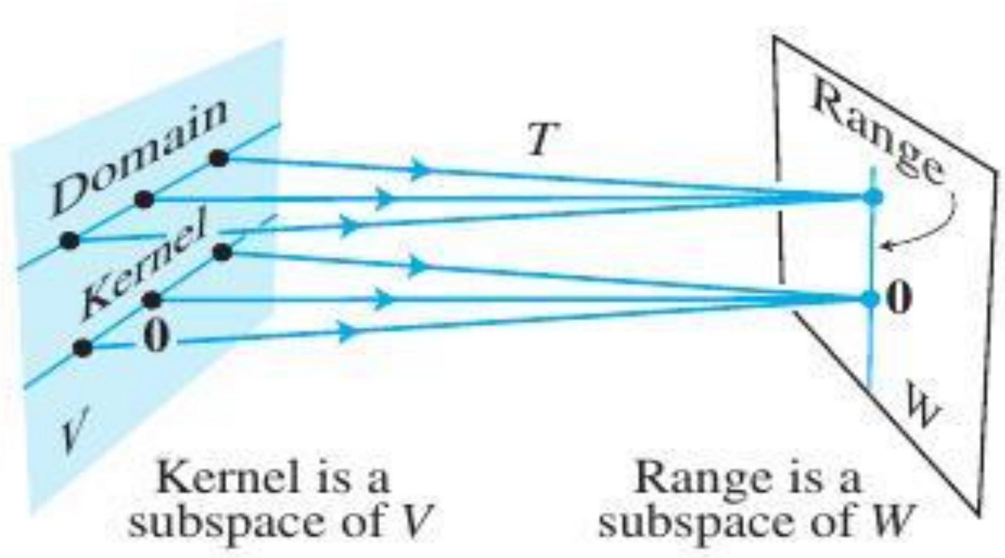


KERNEL AND RANGE

- **Theorem:** Let $T: V \to W, x \mapsto T(x)$ be a linear transformation.

  1.) The **kernel** $Nul(T)$ is a subspace of V.

  2.) The **range** $T(V)$ is a subspace of W.



Proof:

KERNEL AND RANGE

NULL SPACE OF A MATRIX

Any matrix A is the standard matrix of a linear transformation T. Hence we can translate the definition of the kernel or null space into matrix notation:

Definition: The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $Ax = 0$:

$$Nul\ A = \{x \text{ in } \mathbb{R}^n, \text{such that } Ax = 0\}.$$

Theorem 2: The **null space** of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. It is the set of all **solutions** to the system $\boldsymbol{Ax = 0}$.

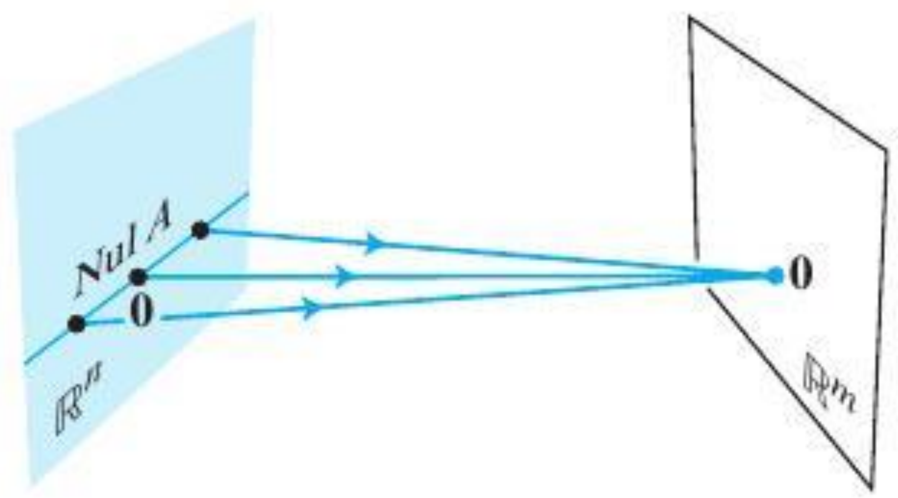
Proof: There is nothing to prove as this follows from the general case.

NULL SPACE OF A MATRIX

Note: (Implicit description of Nul A)

We say that Nul A is defined **implicitly**, because it is defined by a condition that must be checked.

- **Nul A** is the set of solutions of the equation $Ax = 0$. This gives an **explicit** description of Nul A.



NULL SPACE OF A MATRIX

Example: Find a spanning set for the null space of the matrix $Ax = 0$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

NULL SPACE OF A MATRIX

- The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$ with x_2, x_4, and x_5 free.
- Transforming this into a parametric description we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2\mathrm{u} + x_4\mathrm{v} + x_5\mathrm{w} \quad \mathbf{(1)}$$

u v w

NULL SPACE OF A MATRIX

- Every linear combination of **u**, **v**, and **w** is an element of Nul A.
- Thus {**u**, **v**, **w**} is a spanning set for Nul A.

1. The spanning set produced by this method is **automatically linearly independent** because for each free variable we get a row with only one 1 and otherwise 0s. Each time at a different position.

2. When Nul A contains nonzero vectors, the **number of vectors** in the spanning set for Nul A equals the **number of free variables** in the equation $Ax = 0$.

COLUMN SPACE OF A MATRIX

Translating the definition of the range into matrix notation, we get:

- **Definition:** The **column space** Col A of an $m \times n$ matrix A, is the set of all linear combinations of the columns of A.

If $A = [a_1, a_2, \dots, a_n]$, then

Col A = Span$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\}$ or

Col A = $\{b$ in $\mathbb{R}^m$, where $b = Ax$ for some x in $\mathbb{R}^n\}$.

Theorem 3: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$. It is the **range** of the linear transformation $T: x \mapsto T(x) = Ax$.

Proof: $A = [T(e_1), T(e_2), \dots, T(e_n)]$ is the standard matrix of a linear transformation T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$, $Col\ A = T(\mathbb{R}^n)$. We have already proven the more general case.

COLUMN SPACE OF A MATRIX

Example: Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $\mathrm{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathrm{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

1.) Determine if $\mathbf{u}$ is in Nul A. Could $\mathbf{u}$ be in Col A?
2.) Determine if $\mathbf{v}$ is in Col A. Could $\mathbf{v}$ be in Nul A?

COLUMN SPACE OF A MATRIX