Math 22 –
Linear Algebra and its applications

- Lecture 13 -

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GENERAL INFORMATION

- **Office hours:** **Tu 1-3 pm**, Th, Sun 2-4 pm in **KH 229**

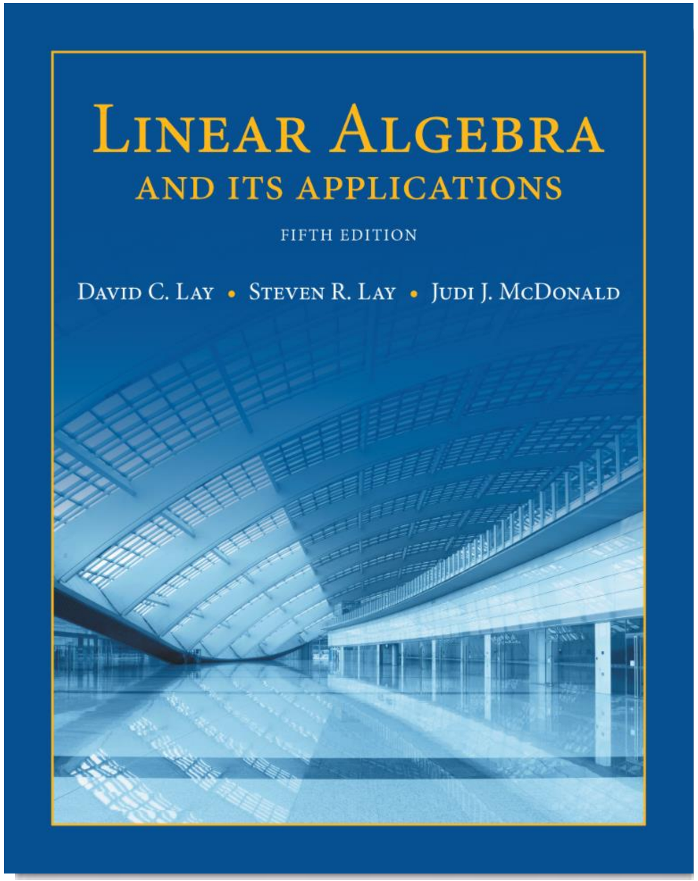
- **Tutorial:** **Tu**, Th, Sun **7-9 pm** in **KH 105**

- **Homework 4:** due **Wednesday** at **4 pm** outside **KH 008**. Please divide into the parts **A, B, C** and **D** and **write your name** on each part.

4 Vector Spaces

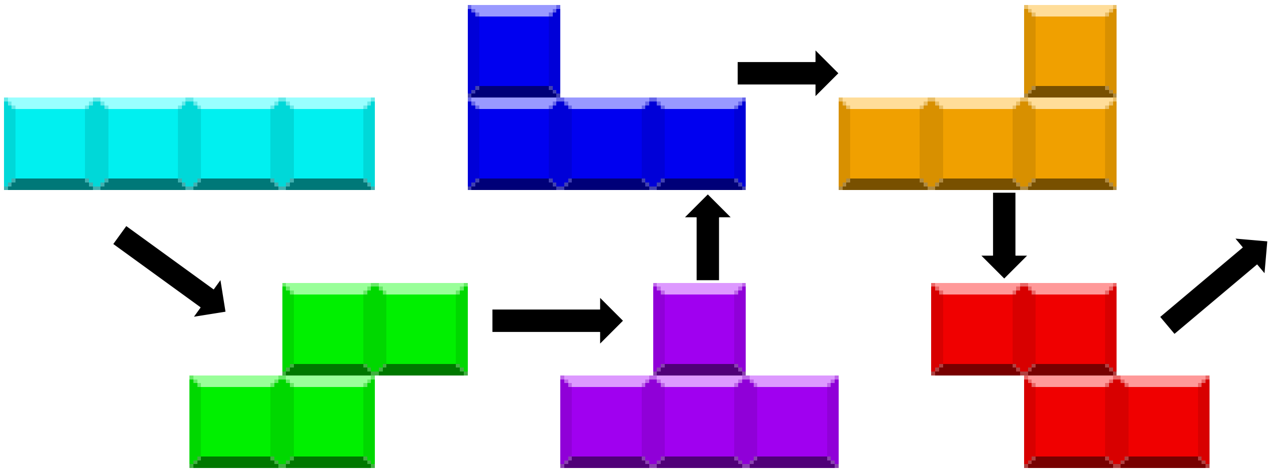
4.1

VECTOR SPACES AND SUBSPACES



Summary:

1.) A **vector space** generalizes the notion of a **coordinate system**. Many **unexpected spaces** are **vector spaces**.



2.) In a **vector space** we can do **linear algebra** as **usual.**

VECTOR SPACES

- **Definition:** A **vector space** is a nonempty set V of objects, called **vectors**, on which are defined two operations, called **addition** and **multiplication by scalars** subject to the ten axioms below.

These axioms must hold for all vectors **u**, **v**, **w** in V and scalars c,d in $\mathbb{R}$.

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(\mathrm{c} + \mathrm{d})\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Consequences:

1.) These axioms imply that the zero vector **0** is unique. The vector **-u**, called the **negative** of **u** is unique for each **u** in V.

2.) For each vector **u** in V and scalar c in $\mathbb{R}$ we have:

a) $0\mathbf{u}=\mathbf{0}$ and $c\mathbf{0}=\mathbf{0}$

b) $-\mathbf{u} = (-1)\mathbf{u}$

Proof: 1.)

VECTOR SPACES

Note 1: 1.) To show that a space **is a vector space** we have to **check** that **all ten axioms** are satisfied.

2.) To show that a space **is not a vector space** we have to show that **an axiom fails** to be true for the space in question.

Note 2: A rule of thumb is that

1.) if addition and scalar multiplication is defined **entrywise**, i.e. there **is no interaction** between the different entries **and**

2.) these **two operations** are defined in the **usual way** **and**

3.) the **value in each entry** can be **any number in** $\mathbb{R}$

then we have a **vector space**.

Examples:

VECTOR SPACES

VECTOR SPACES

SUBSPACES

Definition: A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector $\mathbf{0}$ of V is in H.

b. For each vector $\mathbf{u}$ and $\mathbf{v}$ in H, $\mathbf{u}+\mathbf{v}$ is in H.

c. For each $\mathbf{u}$ in H and each number c, $c\mathbf{u}$ is in H.

Properties (a), (b), and (c) guarantee that a H of V is itself a **vector space**, under the linear space operations already defined in V.

Note: 1.) To show that a subset of a vector space is a subspace, we have to **verify all three conditions**.

2.) To show that a subset of a vector space is **not** a subspace, it is sufficient to show that **one** of the conditions **is not satisfied**.

Examples:

LINEAR SPACE - PROPERTIES

- **Definition: A linear combination** refers to any sum of scalar multiples of vectors

$$c_1\mathbf{v}_1 + \ldots + c_1\mathbf{v}_p$$

- A **span** Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1,\ldots,\mathbf{v}_p$.

- In the same way we define **linear independence** for a **vector space** and **maps** between **vector spaces**.

- All **theorems** concerning these definitions carry over to general **vector spaces.**

Examples:

CONSTRUCTING SUBSPACES

- **Theorem 1:** If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

- We call Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

- Give any subspace H of V, a **spanning** (or **generating**) set for H is a set $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in H such that

$$H=\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$$

Examples: