Math 22 –
Linear Algebra and its applications

- Lecture 12 -

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GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, Th, **Sun 2-4 pm** in KH 229

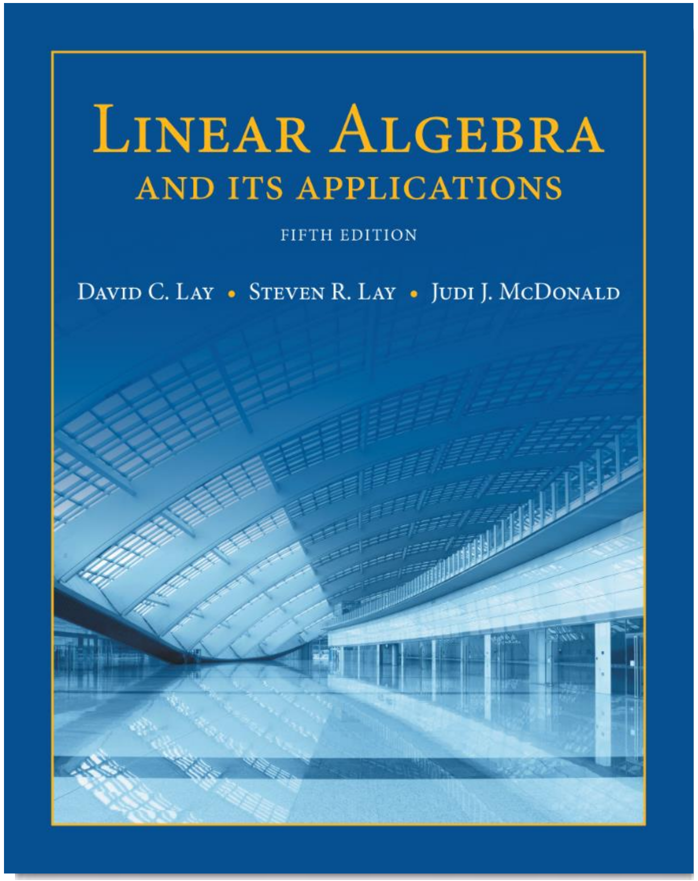
- **Tutorial:** Tu, Th, **Sun 7-9 pm** in KH 105

- **Homework 4:** due **Wednesday** at **4 pm** outside **KH 008**. Please divide into the parts **A, B, C** and **D** and **write your name** on each part.

3 DETERMINANTS

3.1

DETERMINANTS AND THEIR PROPERTIES



- **Summary**:

1.) The **determinant measures** how much the linear map **transforms** the **volume of an object**.

2.) There are **different ways** to **calculate** the **determinant**. One is to use the **row reduction** algorithm.

GEOMETRIC INTERPRETATION

GEOMETRIC INTERPRETATION

DETERMINANTS

- **Definition**: Let $A = [a_{ij}]$ be an $n \times n$ matrix. We call $\boldsymbol{A_{ij}}$ the $n-1 \times n-1$ **submatrix** which we obtain by **deleting** the **i-th row** and **j-th column**.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}.$$

$$\boldsymbol{A_{21}} = \begin{bmatrix} & a_{1,2} & \cdots & a_{1,n} \\ & & & \\ & \vdots & \ddots & \vdots \\ & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}.$$

DETERMINANTS

- **Definition**: For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries a_{11}, a_{12} , . . . , a_{1n} are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} det A_{12} + \cdots + (-1)^{1+n} a_{1n} det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} det A_{1j}$$

We can expand the definition to any square matrix by setting **det(a)=a** for a *1x1* matrix, which is simply an entry.

In two dimensions this reduces to the usual formula.

DETERMINANTS

Example: Compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

DETERMINANTS

- **Defintion:** Given $A = [a_{ij}]$, the **(*i, j*)-cofactor** of A is the number C_{ij} given by

$$\boxed{C_{ij} = (-1)^{i\,+\,j}\, det A_{ij}}\,.$$

The **sign of the cofactor** can be read from the **sign matrix**:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix} \qquad \boldsymbol{A_{21}} = \begin{bmatrix} & a_{1,2} & \cdots & a_{1,n} \\ & & & \\ & \vdots & \ddots & \vdots \\ & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}.$$

Example: $C_{21} = (-1)^{2+1} det A_{21}$

DETERMINANTS

Theorem: **(Cofactor expansion)**

The determinant of an $n \times n$ matrix A can be computed by a cofactor **across any row** or **down any column**.

The **expansion** across the ***i*-th row** using cofactors is

$$detA = ai_1C_{i1} + ai_2C_{i2} + \cdots + a_{in}C_{in}$$

The **expansion** down the ***j*-th column** is

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

DETERMINANTS

Example: Use a cofactor expansion down the third column to compute det A, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

PROPERTIES OF DETERMINANTS

- **Theorem 1:** If A and B are $n \times n$ matrices, then

$$\det AB = (\det A)(\det B).$$

- **Idea:** The determinant measures the volume deformation.

- **Theorem 2:** If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

- **Proof:** Cofactor expansion by the first column.

- **Theorem 3**: Let A be a square matrix.
  a) If a multiple of one row of A is added to another row to produce a matrix B, then

  $$\mathbf{det}\ \boldsymbol{B} = \mathbf{det}\ \boldsymbol{A}.$$

  b) If two rows of A are interchanged to produce B, then

  $$\mathbf{det}\ \boldsymbol{B} = -\ \mathbf{det}\ \boldsymbol{A} \quad \text{or} \quad \mathbf{det}\ \boldsymbol{A} = -\ \mathbf{det}\ \boldsymbol{B}\ .$$

  c) If one row of A is multiplied by k to produce B, then

  $$\mathbf{det}\ \boldsymbol{B} = \boldsymbol{k} \cdot \mathbf{det}\ \boldsymbol{A} \quad \text{or} \quad \mathbf{det}\ \boldsymbol{A} = \frac{\mathbf{1}}{\boldsymbol{k}} \cdot \mathbf{det}\ \boldsymbol{B}$$

PROPERTIES OF DETERMINANTS

- **Example:** Compute det A, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

  using row reduction and **Theorem 2** and **Theorem 3**.

Solution:

PROPERTIES OF DETERMINANTS

- **Theorem 4:** A square matrix A is invertible if and only if det $A \neq 0$.

- **Proof:**

PROPERTIES OF DETERMINANTS

- **Example:** Show that **det A=0**, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

- **Solution:**

DETERMINANT OF THE TRANSPOSED MATRIX

- **Theorem 5:** If A is a $n \times n$ matrix, then
- $$\boxed{\det A^{T} = \det A}.$$

- **Proof**:

DETERMINANT OF THE TRANSPOSED MATRIX

Example: Verify proof 1.) for **Theorem 5** for $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

DETERMINANT - EXAMPLES

- **Examples:**