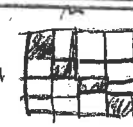
Notation: "$\Rightarrow$" means implies, "$\Leftrightarrow$" means "is equivalent"

Setup: let $\boxed{T}: \mathbb{R}^{\boxed{n}} \to \mathbb{R}^{\boxed{m}}$ be a linear transformation and $\boxed{A}$ be the corresponding standard matrix. Note that $\boxed{n}$ does $\boxed{\text{not}}$ have to be equal to $\boxed{m}$.

Viewpoint	Theorem I: (one-to-one)	Theorem II: (onto)
Geometric	1.) T is one-to-one	I) T is onto
Linear equations	2.) • $Ax=b$ has at most 1 solution • $Ax=0$ has only the trivial solution $x=0$	II) $Ax=b$ has always a solution.
Linear dependence/ Span	3.) The columns of A are linearly independent.	III) The columns of A span $\mathbb{R}^m$
Pivots	$\boxed{4.)}$ A has a pivot in every $\boxed{\text{column}}$	$\boxed{\text{IV}}$) A has a pivot in every $\boxed{\text{row}}$
Matrix equations	5.) There is an $\boxed{n \times m}$ matrix D, such that $D \cdot A = I_{\boxed{n}}$	V) There is an $\boxed{n \times m}$ matrix C, such that $A\,C = I_{\boxed{m}}$

$\Downarrow$ $\boxed{n \leq m}$ $\qquad\qquad$ $\Downarrow$ $\boxed{n \geq m}$

In the $\boxed{\text{special case}}$, where $\boxed{n = m}$, we have 4.) $\Leftrightarrow$ IV). (NOT true in general)

and $\boxed{\text{all}}$ statements 1.) - 5.) are equivalent to I) - V).  $= A$

• Furthermore $\boxed{\text{in this case}}$ $C = D = A^{-1}$ is the inverse of A.

This proves the Inverse Matrix Theorem (Theorem 8)

proof: $\boxed{\text{Theorem I} \quad 5.) \Rightarrow 2.)}$

We know: There is a matrix D, such that $\boxed{D\cdot A = I_n}$

To show: $Ax=0$ has only the trivial solution $x=0$.

given x in $\mathbb{R}^n$, such that $Ax=0$.

Then $D\cdot A . x = D\cdot 0 = 0$

But $\underbrace{DAx}_{=} = I_n x = x$, hence $x=0$.

Hence $x=0$ is the only solution.

proof: $\boxed{\text{Theorem I} \quad 4.) \Rightarrow 5.)}$

We know: A has a pivot in every column. (Recall that $n \le m$)

Translating the row reduction into elementary matrices there is a product $E_p \cdots E_1 = E$ of elem. matrices such that

$E\cdot A =$  $= \tilde{I}$

Set $D = \tilde{I}^T\cdot E$. Then $D\cdot A = \tilde{I}^T E A = \tilde{I}^T\tilde{I} = I_n$

proof: $\boxed{\text{Theorem II} \quad \text{III}) \Rightarrow \text{V})}$

We know: $Ax=b$ has always a solution. Hence there are $y_1, \ldots, y_m$ in $\mathbb{R}^n$, such that $A\cdot y_k = e_k$ (e_k basis vector in $\mathbb{R}^m$)

Writing $C = [y_1, \ldots, y_m]$ in matrix form, this implies

$$A\cdot C = I_m$$

proof: $\boxed{\text{Theorem II} \quad \text{V}) \Rightarrow \text{III})}$

We know: $A\cdot C = I_m$.

To show: For any b in $\mathbb{R}^m$ there is an x in $\mathbb{R}^n$, such that

$$Ax=b.$$

Set $x = C\cdot b$. Then $A(Cb) = (AC)b = I_m b = b$.