Math 22 –
Linear Algebra and its applications

- Lecture 11 -

Instructor: Bjoern Muetzel

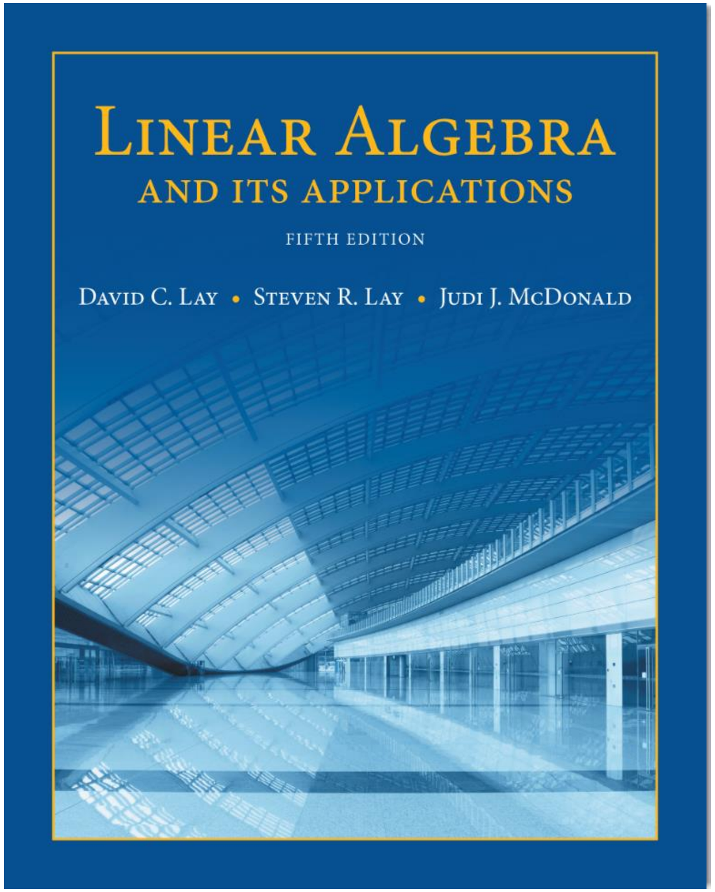
GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, **Th**, Sun 2-4 pm in KH 229
- **Tutorial:** Tu, **Th**, Sun **7-9 pm** in KH 105
- **Homework 3:** due **Wednesday** at **4 pm** outside **KH 008**. Please divide into the parts **A, B, C** and **D** and **write your name** on each part.
- **Saturday / Sunday:** Group meetings for the project

2 Matrix Algebra

2.3

CHARACTERIZATIONS OF INVERTIBLE MATRICES



- <u>**Summary**</u>**:**

  1.) If a **linear transformation is invertible** then we can **undo** its effect on the space using the **inverse transformation**.

  2.) The **inverse matrix theorem unites** the **different viewpoints** of a transformation that is both **one-to-one** and **onto**.

GEOMETRIC INTERPRETATION

INVERTIBLE LINEAR TRANSFORMATIONS

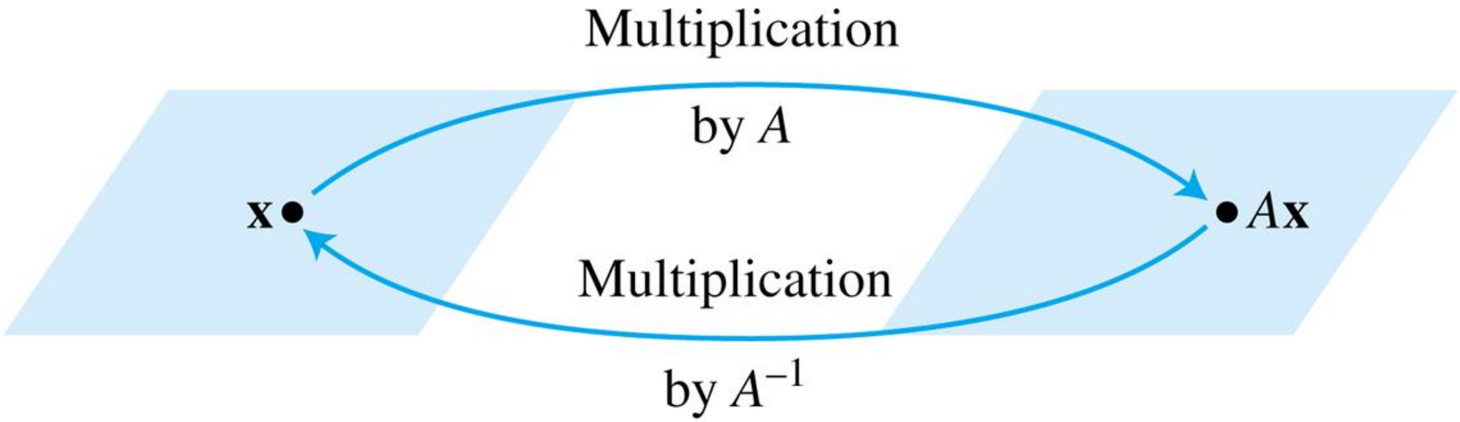
- **Definition:** A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(x)) = x \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (1)$$

$$T(S(x)) = x \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (2)$$

INVERTIBLE LINEAR TRANSFORMATIONS

- **Theorem 9:** Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the **standard matrix** for T. Then T is invertible if and only if A is an invertible matrix. In that case, the inverse linear transformation S given by $S(x) = A^{-1}x$.



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

THE INVERTIBLE MATRIX THEOREM

- **Theorem 8:** Let A be a **square $n \times n$ matrix**. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.
  a. A is an invertible matrix.
  b. A is row equivalent to the $n \times n$ identity matrix.
  c. A has n pivot positions.
  d. The equation $Ax = 0$ has only the trivial solution.
  f. The columns of A form a linearly independent set. The linear transformation $x \rightarrow Ax$ is one-to-one.
  g. The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$.
  h. The columns of A span $\mathbb{R}^n$.
  i. The linear transformation $x \rightarrow Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
  j. There is an $n \times n$ matrix C such that $CA = I_n$.
  k. There is an $n \times n$ matrix D such that $AD = I_n$.
  l. A^T is an invertible matrix.

THE INVERTIBLE MATRIX THEOREM

THE INVERTIBLE MATRIX THEOREM

- **Theorem 8** could also be written as

  "The equation $Ax = b$ has a *unique* solution for each **b** in $\mathbb{R}^n$."

- **<u>Consequence:</u>** Let A and B be square matrices. If $AB = I_n$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

- The **Invertible Matrix Theorem** divides the set of all $n \times n$ matrices into **two disjoint classes**:

  1.) the **invertible** or **nonsingular** matrices, and

  2.) the **noninvertible** or **singular** matrices.

THE INVERTIBLE MATRIX THEOREM

- Each statement of **Theorem 8** describes a property of every $n \times n$ invertible matrix.
- The ***negation*** of a statement in the theorem describes a property of every $n \times n$ singular matrix.

- **Example:** An $n \times n$ singular matrix is *not* row equivalent to I_n, does *not* have n pivot position, and has linearly *dependent* columns.

THE INVERTIBLE MATRIX THEOREM

- **Example:** Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

- **Solution:**