Math 22 – Linear Algebra and its applications

- Lecture 10 -

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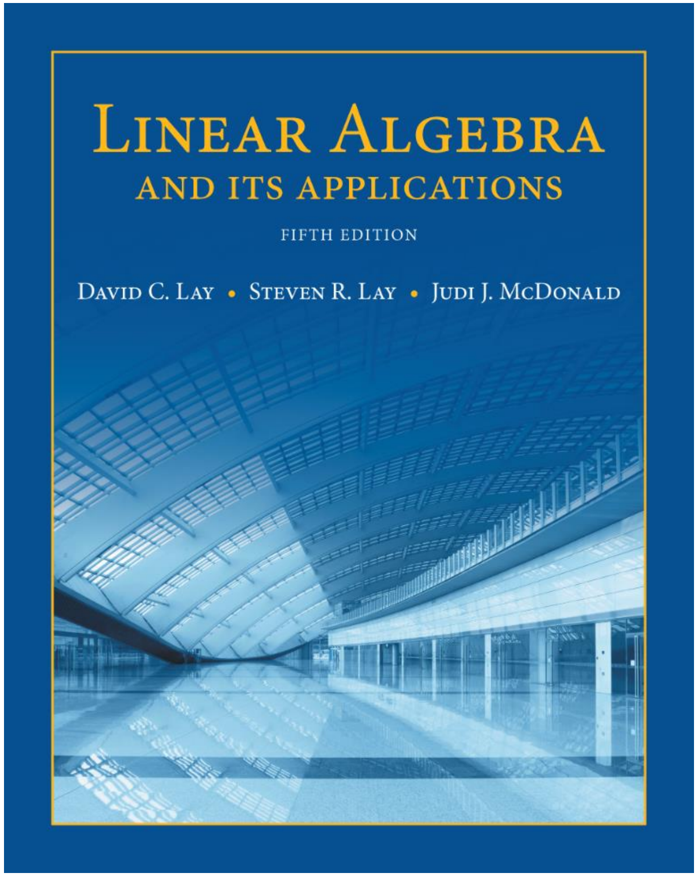
GENERAL INFORMATION

- **Office hours: Tu 1-3 pm**, Th, Sun 2-4 pm in KH 229
- **Tutorial: Tu**, Th, Sun **7-9 pm** in KH 105
- **Homework 3:** due **Wednesday** at **4 pm** outside **KH 008**. Please divide into the parts **A, B, C** and **D** and **write your name** on each part.
- **Midterm 1**: **today Oct 7** from **4-6 pm** in **Carpenter 013**

2 Matrix Algebra

2.2

THE INVERSE OF A MATRIX



- **Summary**:

1.) An $\boldsymbol{n \times n}$ **matrix** has an **inverse** if and only if the corres-ponding **linear transformation** is **both one-to-one and onto**.

2.) We can **find** the **inverse** of a matrix using the **row reduction algorithm**.

INVERSE OF A MATRIX

- **Definition 1:** An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$\boxed{AC = I_n = CA}$$

In this case, C is called the **inverse** of A and we write $\boldsymbol{C = A^{-1}}$.

- **Geometric interpretation:** If $\mathbf{x}$ in $\mathbb{R}^n$ is a vector and

$$S(\mathbf{x}) = A\mathbf{x} \text{ and } T(\mathbf{x}) = C\mathbf{x},$$

are the corresponding linear transformations, this means that

$$S \circ T(\mathbf{x}) = A(C\,\mathbf{x}) = (AC)\mathbf{x} = I_n\mathbf{x} = \mathbf{x} \quad \text{and} \quad T \circ S(\mathbf{x}) = \mathbf{x}$$

This means that S "**undoes**" the **effect** of T and **vice versa**.

Examples: What is the inverse of a rotation, shear and reflection?

Does a projection have an inverse?

INVERSE OF A MATRIX

- **Theorem 4 (Inverse of a 2×2 matrix)** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
  If $\boxed{ad - bc \neq 0}$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

  If $\boxed{ad - bc = 0}$, then A is **not invertible**.

- The quantity $\mathbf{det}(\boldsymbol{A}) = ad - bc$ is called the **determinant** of A.
- This theorem says that a 2×2 matrix A is invertible if and only if $\det(A) \neq 0$.

- **Proof (for det(A)):** Bring A into echelon form and check.

Examples:

INVERSE OF A MATRIX

- **Theorem 5*:** Let A be an $n \times n$ matrix.

  1.) If A is invertible then for each **b** in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

  2.) If for each **b** in $\mathbb{R}^n$, the equation $Ax = b$ has a unique solution, then A is invertible.

- **Proof: 1.)** Let A be an invertible matrix.

Theorem: Let A be an $n \times n$ matrix, then A is invertible **if and only if**

1.) For each **b** in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

2.) The corresponding linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is both one-to-one and onto.

3.) A has a **pivot** in **every row** and in **every column**.

INVERSE OF A MATRIX

Theorem 6: **a.** If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and its inverse is

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is

$$(A^T)^{-1} = (A^{-1})^T$$

Proof:

MATRIX INVERSION - ALGORITHM

Summary: We can **find** the i**nverse** of a matrix using the **row reduction algorithm**.

MATRIX INVERSION - ALGORITHM

- **Theorem 7:** An $n \times n$ matrix A is invertible if and only if A is **row equivalent** to I_n, shortly $\boldsymbol{A \sim I_n}$.

  In this case we obtain A^{-1} by applying the row reduction algorithm to the matrix $[A, I_n]$ which transforms into $[I_n, A^{-1}]$.

- **Proof:** 1.) Suppose that A is invertible.

  We have seen that **Theorem 5*.1.)** implies that A has a pivot position in every row and column.

  This implies that the pivot positions must be on the diagonal, and the **reduced echelon form of A is I_n**. Hence $A \sim I_n$.

2.) Suppose that $A \sim I_n$,i.e A can be row reduced to I_n.

Idea: Calculate A^{-1}:

Hence the equation $Ax=b$ has a unique solution for each **b** in $\mathbb{R}^n$.

Let $\{e_1, e_2, \dots, e_n\}$ in $\mathbb{R}^n$ be the standard basis of $\mathbb{R}^n$.

As a solution exists we can solve the following equations:

$$Ay_1 = e_1, Ay_2 = e_2, \dots, Ay_n = e_n.$$

But this implies for the matrix $B = [y_1, y_2, \dots, y_n]$ that

$$AB = [e_1, e_2, \dots, e_n] = I_n \qquad (1)$$

The row reduction algorithm to solve the equation $Ay_k = e_k$ depends only on A and not on e_k. Hence we can find the solutions simultaneously, applying the algorithm to the matrix

$$[A, I_n]$$

MATRIX INVERSION - ALGORITHM

This matrix reduces to $[I_n, C\ ]$. But translating this back into a system of linear equations we get $[y_1, y_2, \ldots, y_n] = C$.

Hence $C = [y_1, y_2, \ldots, y_n] = B$ and $[A, I_n]$ row reduces to $[I_n, B\ ]$. This implies that $[A, I_n] \sim [I_n, B\ ]$ ($\sim$ means row equivalent).

The row reduction algorithm does not depend on the relative position of A and I_n. Applying the same steps we get $[I_n, A] \sim [B, I_n\ ]$.

Applying the inverse steps of the row reduction backwards we can transform $[B, I_n\ ]$ into $[I_n, A]$. But this means that $AB = I_n$.

With (1) we get $BA = AB = I_n$ and $\mathbf{B} = \boldsymbol{A}^{-1}$.

MATRIX INVERSION - ALGORITHM

- **Example:** Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.
- **Solution:**

MATRIX INVERSION - ALGORITHM

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- **Theorem 7** shows, since $A \sim I_3$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

- Now, check the final answer.

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ELEMENTARY MATRICES

- **Definition:** An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

- **Example:**

Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

1.) Compute E_1A, E_2A, and E_3A, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$.

2.) Describe the effect of these multiplications on A.

3.) Find the inverse matrices of E_1, E_2, and E_3.

ELEMENTARY MATRICES

ELEMENTARY MATRICES

Summary:

- If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I_m.

ELEMENTARY MATRICES

- **Summary**
- Applying this to the problem of finding the inverse matrix we get:
- Each step of the row reduction of A corresponds to left-multiplication by an elementary matrix, there exist elementary matrices $E_1, \ldots, E_p$ such that

$$A \sim E_1 A \sim E_2(E_1 A) \sim \ldots \sim E_p(E_{p-1} \ldots E_1 A) = I_n$$

That is, $E_p \ldots E_1 A = I_n$.

Since the product $E_p \ldots E_1$ of invertible matrices is invertible, this leads to $(E_p \ldots E_1)^{-1}(E_p \ldots E_1)A = (E_p \ldots E_1)^{-1} I_n$ or

$$A = (E_p \ldots E_1)^{-1}$$

Hence A is invertible and $A^{-1} = \left[(E_p \ldots E_1)^{-1}\right]^{-1} = E_p \ldots E_1$.