Math 22: Linear Algebra
Fall 2019 - Homework 9

not graded

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1. Consider the following linear transformations (see **Example 3** and **Tables 1 & 3** on pages 73 - 76 of the book).
   Determine whether the corresponding standard matrix of the given transformation has a real eigenvalue. If so, state **one** eigenvalue and a corresponding eigenvector. Justify your answer.

   a) $T : \mathbb{R}^2 \to \mathbb{R}^2, \mathbf{v} \mapsto T(\mathbf{v}) = 3\mathbf{v}$.

   b) $S : \mathbb{R}^2 \to \mathbb{R}^2$, where S is a reflection with respect to the line $\{c \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, where $c \in \mathbb{R}\}$.

   c) A horizontal shear in $\mathbb{R}^2$.

   d) $R(\frac{\pi}{2}) : \mathbb{R}^2 \to \mathbb{R}^2$, where $R(\frac{\pi}{2})$ is the counterclockwise rotation about the origin with angle $\frac{\pi}{2}$.

2. Find the characteristic polynomial of the following matrices, then determine the eigenvalues and their multiplicity. Justify your answer.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & 2 & 1 \\ 0 & 2 & 0 \\ 4 & 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & -4 & -2 & 4 \\ 0 & -2 & 9 & 2 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & -\frac{1}{2} & 4 \end{bmatrix}.$$

3. Let A be the 2×2 matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

   a) Find the eigenvalues of A.

   b) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

   c) Diagonalize the matrix A.

   d) Calculate A^7.

   e) Diagonalize the inverse matrix A^{-1} of A.

   Hint: For d) and e) you can use the information from c).

4. **(Fibonacci sequence) (C)** Let $(x_k)_{k \in \mathbb{N}}$ be the Fibonacci sequence given by the recursive formula

$$x_0 = 0\ ,\ x_1 = 1\ ,\ \boxed{x_{k+2} = x_{k+1} + x_k} \text{ for all } k \geq 0.$$

   So we obtain the sequence

$$0\,,\,1\,,\,1\,,\,2\,,\,3\,,\,5\,,\,8\,,\,13\,,\,21\,,\,34\,,\,55\ldots$$

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We have seen that the relation between the elements of the sequence can be described using a matrix:

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} \qquad (1)$$

Note: For this problem you **can** use a computer algebra program like Wolfram Alpha to find the solutions.

a) Watch the videos **Vi Hart - Fibonacci sequence in plants** on the *Resources* page of our website.

b) In the vector form, this can be seen as a discrete dynamical system. Draw the first vectors $\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$, for $k = 0, 1, 2, 3, 4, 5, 6$ in the plane $\mathbb{R}^2$. What do you notice.

c) Find the eigenvalues of A.

d) Diagonalize the matrix A.

e) It follows from (1) that $\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = A^k \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. Use part d) to find a simple formula for A^k and then find a formula for x_k.

5. **(Markov chains) (C)** Draw your own (imaginary) mood diagram (see **Lecture 28**) and find the steady state.
   Note: For this problem you **can** use a computer algebra program like Wolfram Alpha to find the solutions.